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Hedging: imperfect correlation, timely information and stochastic volatility

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Hedging: Imperfect Correlation, Timely Information and Stochastic Volatility

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Summary

We present three somewhat related problems in mathematical finance. Firstly, consider a trading wealth generated from investing in a geometric-Brownian asset and consider a geometric-Brownian contingent claim. We investigate the problem of finding the trading strategy which maximises the probability of the wealth exceeding the claim value at some horizon. This is often called the quantile hedging problem.

We show that the general imperfectly correlated case with drift can be reduced to a problem with zero correlation but modified parameters. We obtain the Hamilton-Jacobi-Bellman equation for the problem which we solve using a combination of the Crank-Nicholson scheme and policy improvement.

Using barrier option methods we reduce an American version of this problem to a European problem. We compare receiving information on the claim value continuously to receiving it instantaneously.

Our second problem is that of finding the best observation times for approximating the value of Asian options. We consider Brownian and geometric-Brownian assets. In the latter case we find a symmetry related to asset drift.

Finally we consider utility maximisation under stochastic volatility. For constant relative risk aversion we obtain the value function and pricing measure without a Markov assumption. We obtain explicit formulae when the dynamics are geometric-Ornstein-Uhlenbeck.

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Introduction

This thesis is concerned with the problem of an agent who seeks to meet a contingent claim given limited funds. In particular the agent may be only able to trade on an asset partially correlated to the claim, may receive information about the claim according to some restrictive schedule or may find that the volatility of the contingent claim is stochastic.

Suppose we have a trading wealth X^{ϕ} which evolves according to the strategy ϕ for trading in some asset and a contingent claim C that we shall be obliged to meet at some time horizon T. We shall be concerned with maximising the chance of meeting the claim, that is with finding $\sup_{\phi} \mathbb{P}(X_T^{\phi} \geq C_T)$. The claim may be correlated to the trading wealth or be independent of it; it may be dependent on some asset or not; it may be revealed at the outset, at the horizon time, continuously up to the horizon time, or instantaneously at some intermediate time.

There are many situations of genuine practical interest where questions of this kind might arise. Trading activity in banks and other financial institutions is often determined by traders trying to reach targets of one kind or another, often linked to their bonus packages. They are not concerned with how much they might fall short of or exceed the target only whether they reach it or not. Similarly portfolio managers and pension fund managers are often concerned with exceeding a certain benchmark (say the FTSE 100 index) by a fixed percentage.

Later we shall be looking at cases where the trading wealth is that from trading on one stock and the contingent claim depends on another stock, which may be only partially correlated with or even independent of the tradeable stock. This is a good way of looking at a situation where a financial institution has entered into an agreement to pay a claim based on some stock but cannot hedge using that stock because regulators prevent it from doing so or there is some liquidity problem in the market for that stock. It is also highly relevant to the problem of hedging a claim based on a basket of many assets. One does not wish to trade in all the assets in the basket because of prohibitive transaction costs, instead one chooses a representative selection from the basket. This selection will be partially correlated with the basket.

These ideas have applications in wider contexts. Kelly [35] applies ideas of gambling to the concept of information rate in communication theory. Turnbull [51] explains how such ideas can be applied to problems of controlling water level in a dam with bankruptcy corresponding to an empty reservoir.

A number of academics and practitioners have looked at problems in this area. Kulldorff [36] controls the volatility of a geometric-Brownian trading wealth to reach a constant claim. He considers both discrete and continuous time. The former is essentially a linear programming problem that will not concern us. In the latter, he obtains a partial differential equation for the value function, that is the probability of meeting the claim given a particular wealth and time to the horizon.

Browne [6] also has a constant claim. He has n geometric-Brownian assets and also obtains a partial differential equation for his value function. He analyses the region in which borrowing takes place, that is wealth and time-to-go pairs for which the holding in cash is negative (though total portfolio value will always be positive).

Föllmer and Leukert [16] have a claim revealed at horizon time and their tradeable asset is a general semimartingale. They consider both complete and incomplete markets. They reduce the problem to that of finding the maximal success set and then apply the Neyman-Pearson Lemma. Our analysis is different in that the incompleteness is introduced in a different way.

Spivak and Cvitanić [49] also consider the problem where the claim is revealed at the horizon time but their asset is geometric-Brownian. They consider the dual problem and find that they should meet the claim when it is small and ensure that they are left with nothing when the claim is large.

The approach we shall take will be structured as follows. In Chapter 1 we directly analyse claims revealed instantaneously in time, that is at the outset (time 0) or at the horizon time (time T) both in the complete market and in the case of an independent claim. We also describe the methods used in earlier work and explain how their results fit in with ours.

Chapter 2 will have claims continuously revealed and there will be partial correlation between our tradeable asset and our contingent claim. Here we will restrict ourselves to geometric-Brownian processes, as Kulldorff [36], Browne [6] and Spivak and Cvitanić [49] did, though in each case their claim was revealed at horizon time. That is we have a claim with value C and a traded asset with price P given by

$$\frac{dC_t}{C_t} = \sigma_t dW_t + \nu_t dt, \quad C_0 = c_0 \tag{1}$$

$$\frac{dP_t}{P_t} = \eta_t \left\{ \rho_t dW_t + \sqrt{1-\rho_t^2} dB_t + \mu_t dt, \right\} \tag{2}$$

(where B and W are independent Brownian motions) with a strategy ϕ_t chosen by the agent determining how the trading wealth evolves,

$$\frac{dX_t^\phi}{X_t^\phi} = \phi_t \frac{dP_t}{P_t}, \quad X_0 = x_0.$$

We will first derive in Section 2.2 a general partial differential equation for this class of problems. In Section 2.3 we show the equivalence of optimising the success probability and optimising the expected ratio of wealth to claim, $U_T = X_T / C_T$. That is we show $\sup_\phi \mathbb{P}(X_T^\phi \geq C_T) = \sup_\phi \mathbb{E}[U_T \wedge 1]$. Such wealth or success ratios feature in Föllmer and Leukert [16]

In Section 2.4 we shall consider the complete market case. We obtain explicit solutions and confirm that they satisfy the general partial differential equation of Section 2.2. We shall show that a transformation reduces the general case to either the case with a zero-drift asset or to the case with a constant claim.

Section 2.5 introduces an independent asset. In the case of a zero-drift independent asset we are able to conjecture an optimal policy and show that its value function satisfies the required equation.

There are a number of related problems of interest that we look at in Chapter 3. The problem with multiple assets can be reduced to the single asset problem we have considered. We show how this is done in Section 3.1. A number of authors, including Föllmer and Leukert [17], have considered the problem of minimising the expected shortfall. In Section 3.2 we show how in our geometric-Brownian context this can be reduced to the quantile-hedging problem.

In Chapter 4 we turn to a numerical approach for solving the independent asset problem. In Section 4.2, we solve the zero-drift case numerically as the comparison between exact and numerical solutions is useful later when we cannot obtain an exact solution. To solve numerically we require a transformation of variables and a careful choice of boundary conditions.

Section 4.3 considers the case of an independent asset with non-zero drift where we find we need to use a policy improvement algorithm as the partial differential equation to be solved otherwise is non-linear. We consider how to ensure stability of the calculated policy and how to make a suitable choice of boundary conditions. We interpret the results obtained in Section 4.4. In Section 4.5 we briefly consider the effect of a constraint on selling stock short.

The key point is that we have our claim *revealed continuously* and we have *imperfect correlation* between tradeable asset and contingent claim.

In Chapter 5 we consider a variant on the quantile hedging problem where if at any time one has sufficient wealth one can call in the contingent claim. This is essentially a barrier option problem. It was posed by Karatzas [33]. We find that the value improvement in this American problem is very similar to that found in the European case.

Related to the issue of whether one can call in the claim before maturity is how much information one receives about the claim before maturity. We consider, in Chapter 6, how useful it is to observe the claim evolve through time compared with having it revealed all at one instant. In addition to varying the time at which the claim is revealed we are able to obtain results on how the success probability depends on the timing of volatility for a continuously-revealed claim. We do this for the quantile-hedging case in Section 6.1. In Section 6.2, we consider the more tractable case of constant relative risk aversion.

In addition to the quantile hedging problem we look at two related problems. We develop ideas of *timely information* further in Chapter 7. We consider

$$\int_0^T f(X_t)\,dt \rightsquigarrow g\left(\lambda_1, \ldots, \lambda_n; X_{\alpha_1}, \ldots, X_{\alpha_n}\right),$$

that is we approximate the integral on the left by the function on the right. We find the best time points, α_i, to observe the asset X in order to best approximate the value of an Asian option derived from it.

In particular we consider linear approximations of Brownian and geometric-Brownian assets, and minimise the $\mathbb{L}^2$-norm of the error. In Section 7.2 we find that for Brownian assets the integration interval is divided in the ratio of two to two to ... to two to one. In the geometric-Brownian case, Section 7.3, we obtain a symmetry between different asset drifts. We are able to explain this by a change of numéraire, in a similar fashion to the symmetry between floating-strike and fixed-strike Asian options demonstrated by Henderson and Wojakowski [22].

Finally, in Chapter 8 we consider maximising the utility of constant relative risk aversion when the tradeable asset has *stochastic volatility*. That is our utility function is

$$U(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } R \neq 1 \\ \log x & \text{if } R = 1 \end{cases}.$$

Our price process has dynamics,

$$\frac{dP_t}{P_t} = Y_t \left(dW_t^1 + \lambda_t \, dt \right),$$

where the volatility Y is given by

$$dY_t = a_t \, dW_t^2 + b_t \, dt.$$

The Brownian motions W^1 and W^2 are partially correlated. By following the approach of Hobson [25] we are able to reduce the problem to one of optimisation over a set of measures. Doing this allows us to extend the results of Zariphopoulou [53] by dropping the Markovian assumption. We are also able to obtain an expression for the pricing measure that one would use for utility-indifference pricing. We have explicit results for the case of geometric-Ornstein-Uhlenbeck dynamics, a model first suggested by Scott [48].

Chapter 1

Claims Revealed Instantaneously in Time

Recall that we are concerned with an agent who has trading wealth X^ϕ dependent on the trading strategy ϕ and who is obliged to meet a contingent claim C at a fixed time horizon. The agent wishes to maximise the probability of meeting the claim, $\mathbb{P}\left(X^\phi \geq C\right)$. There are many practical situations where agents behave in this manner. Trading activity in banks, pension funds and other institutions is often determined by traders or fund managers wishing to meet their target or to exceed a certain benchmark by a fixed percentage.

In this context we are able to retain a high level of generality. For simplicity of exposition we shall work with a discounted tradeable asset and a discounted claim so interest rates will not feature here.

The case of a zero-drift tradeable asset will prove particularly straightforward. When the asset is zero-drift it is a martingale and so the probability measure under which expectations should be taken for pricing is the same as the objective probability measure which we use for our optimisations.

We begin with an original market that is complete. By assumption the wealth X^ϕ from trading in this market is restricted to be non-negative. However, we do not need to assume anything about the dynamics of any underlying assets.

Now we introduce a non-negative contingent claim C that will not be revealed until the final instant, T. This together with the tradeable asset may or may not form a complete market. We consider the two cases separately.

1.1 The Complete Market

This case can be motivated by considering a trader who is concerned with trading so as to outperform the market (possibly by some given proportion). As the market is complete the trader can attain any terminal wealth distribution subject to the budget constraint. That is, the claim that it is desired to reach is dependent on assets that can be traded in but there is only limited wealth available. The completeness of the market is equivalent to the existence of a unique equivalent martingale measure.

Consider first the simplest case where the claim is revealed at the outset, time 0, and the tradeable asset is zero-drift. If $c_0 \leq x_0$ then we do nothing and at time T meet the claim with probability 1. So assume $c_0 > x_0$.

We want to maximise

$$\mathbb{P}(X \geq c_0),$$

subject to $\mathbb{E}[X] = x_0$, recalling that we are in the case where the pricing measure is the objective measure. We cannot afford to always choose $X = c_0$ so we randomize between this and the smallest permitted wealth, $X = 0$. We choose the smallest permitted wealth as doing so will allow us to increase the probability with which we end at $X = c_0$.

That is

$$X^* = \begin{cases} c_0 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}. \tag{1.1}$$

However, $\mathbb{E}[X^*] = x_0$ gives $p = \frac{x_0}{c_0}$. So

$$X^* = \begin{cases} c_0 & \text{with probability } \frac{x_0}{c_0} \\ 0 & \text{with probability } 1-\frac{x_0}{c_0} \end{cases}.$$

We can consider this as replicating $c_0 \mathbb{1}_A$ where $A \in \mathcal{F}_t$ with $\mathbb{P}(A) = x_0/c_0$. We have

$$\mathbb{P}(X^* \geq c_0) = \frac{x_0}{c_0}.$$

Next consider a claim revealed at horizon time. We shall apply a direct Lagrangian approach to choosing our final wealth X to satisfy

$$\sup_X \mathbb{P}(X \geq C),$$

subject to $\tilde{\mathbb{E}}[X] = x_0$.

We assume there is an equivalent martingale measure $\tilde{\mathbb{P}}$ such that X^ϕ is a supermartingale under $\tilde{\mathbb{P}}$ for any ϕ. We shall write $Z = (d\tilde{\mathbb{P}}/d\mathbb{P})|_{\mathcal{F}}$ where $\mathcal{F}$ is the information available at time T. So C is $\mathcal{F}$-measurable. The Lagrangian for the problem is

$$L(X, \lambda) = \int_{\Omega} \left(\mathbb{1}_{\{X(\omega) \geq C(\omega)\}} - \lambda \left(X(\omega) Z(\omega) - x_0 \right) \right) \mathbb{P}(d\omega). \tag{1.2}$$

For $\lambda \geq 0$ the finite maximum occurs at

$$X^* = \begin{cases} 0 & \text{if } C > \frac{1}{Z\lambda} \\ C & \text{if } C \leq \frac{1}{Z\lambda} \end{cases}. \tag{1.3}$$

Taking λ to satisfy the constraint $\mathbb{E}[ZX^*] = x_0$ and using the Lagrangian Sufficiency Theorem we find that the optimal X is

$$X^* = C\mathbb{1}_{\{C \leq c^*\}},$$

where c^* is such that

$$\int_{\{C \leq c^*\}} ZC \, \mathbb{P}(d\omega) = x_0.$$

Figure 1-1 shows an example where $Z = 1$, that is the asset is zero-drift and so the pricing measure is the actual measure. The horizontal axis gives the sample space Ω (or a part of it). The solid line gives the value of the claim over the sample space and the dashed line shows how the terminal wealth value is chosen for each value of the claim.

The diagram for the general Z case would be similar but would be viewed with a suitable change of measure perspective.

1.2 An Independent Claim

Next we take the tradeable asset to be independent of the claim to be hedged. If the value of the asset were revealed at the beginning we would be in our previous case with the claim constant so we have the claim revealed at the end.

Let the sample space be partitioned into a sample space pertaining to X and Z and one pertaining to C, that is let $\Omega = (\Omega' \times \Omega^c)$. We can identify Ω^c with $\mathbb{R}^+$. We assume there is an equivalent martingale measure $\tilde{\mathbb{P}}$ such that X^ϕ is a supermartingale under $\tilde{\mathbb{P}}$ for any ϕ and the law of C is unchanged. Write $Z = \left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right)\Big|_{\mathcal{F}'}$.

We consider first the zero-drift asset case. Suppose C has cumulative distribution

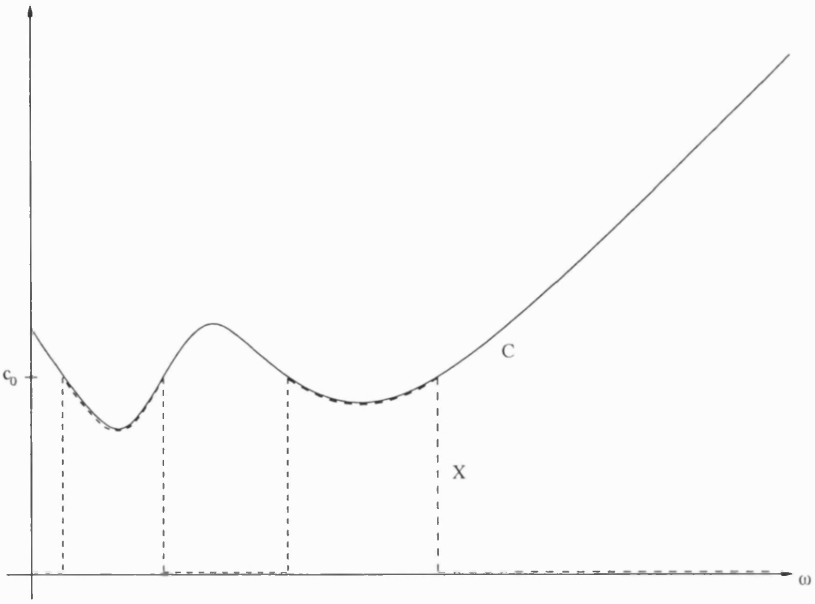

Figure 1-1: The Instantaneously Revealed Claim, Complete Market Case with Martingale Asset

function F_C. If F_C is concave then it lies below any of its tangents. In particular

$$F_C(x) \le \alpha + \beta x,$$

for any $x \ge 0$, where

$$\alpha = F_C(x_0) - x_0 F'_C(x_0)\,, \quad \beta = F'_C(x_0). \tag{1.4}$$

So for any admissible X we have

$$\mathbb{E}[F_C(X)] \le \alpha + \beta \mathbb{E}[X] = F_C(x_0),$$

and so we conclude that

$$X^* = x_0, \tag{1.5}$$

optimises $\mathbb{E}[F_C(X)] = \mathbb{P}(X \ge C)$. That is if F_C is concave it is optimal to do no trading. Intuitively this makes sense as randomising between two points corresponds to a payoff on the line joining them. By concavity this will lie below the distribution function curve.

Consider now claims where F_C is not concave. For a function $f : \mathbb{R}^+ \to \mathbb{R}$ refer to $\tilde{f} = \inf\{g : g \ge f, g\,\text{concave}\}$ as the *concave relaxation* of f. Take $\tilde{F}_C$ to be the concave relaxation of F_C. Let

$$a = \sup\left\{y : y \le x_0, \tilde{F}_C(y) = F_C(y)\right\},$$

and

$$b = \inf\left\{y : y \ge x_0, \tilde{F}_C(y) = F_C(y)\right\},$$

that is take a be the maximal $x \le x_0$ such that $\tilde{F}_C(x) = F_C(x)$ and b the corresponding value on the other side side of x_0. This is depicted in Figure 1-2.

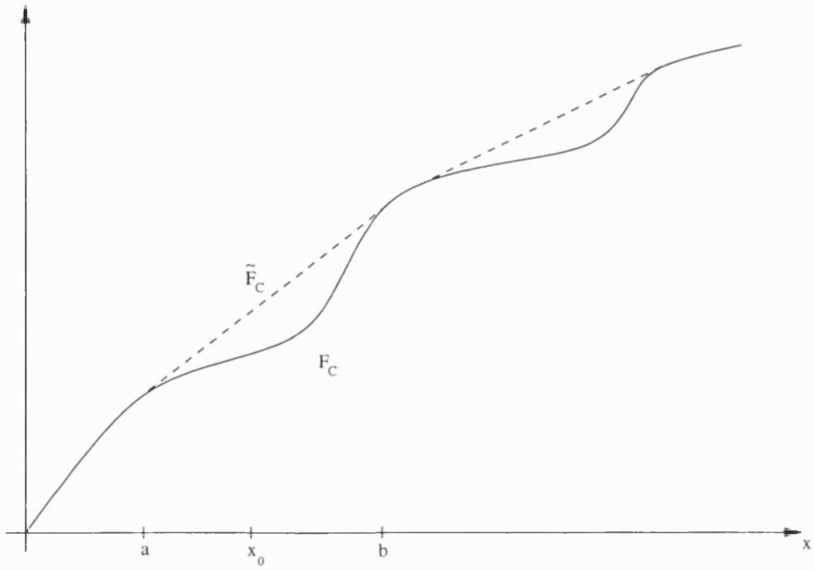

Figure 1-2: Defining the Optimal Solution in the Instantaneously Revealed Claim Independent Asset Case

We conjecture that the optimal solution is to choose a combination of a and b with as much b as we can afford. That is,

$$X^* = \begin{cases} a & \text{with probability } \frac{x_0-b}{a-b} \\ b & \text{with probability } \frac{a-x_0}{a-b} \end{cases}. \tag{1.6}$$

We notice that we randomize as in Spivak and Cvitanić [49] and Föllmer and Leukert [17].

Setting

$$\alpha = \frac{bF_C(a) - aF_C(b)}{b-a} \quad \text{and} \quad \beta = \frac{F_C(b) - F_C(a)}{b-a},$$

we have

$$F_C(a) = \alpha + \beta a\,, \quad F_C(b) = \alpha + \beta b,$$

and

$$F_C(x) \le \alpha + \beta x,$$

for all x by definition of the concave relaxation. We notice that if we take the limit as $b \downarrow a$ we recover the concave case (1.4).

Now for any X we have

$$\mathbb{E}[F_C(X)] \le \alpha + \beta \mathbb{E}[X] = \alpha + \beta x_0,$$

but

$$\mathbb{E}[F_C(X^*)] = \alpha + \beta x_0,$$

so X^* optimises $\mathbb{E}[F_C(X)] = \mathbb{P}(X \ge C)$.

Turning to the case of an asset with non-zero drift, we again first consider claims with concave F_C. The Lagrangian is

$$L(X, \lambda) = \int_{\Omega'} \{F_C(X(\omega)) - \lambda(Z(\omega)X(\omega) - x_0)\} \, \mathbb{P}(d\omega).$$

So we find that

$$X^* = I(\lambda Z),$$

where $I = (F_C')^{-1}$ (F_C' is decreasing but not strictly so; where I is not uniquely defined we will choose the left-continuous version) and λ is such that

$$\int_{\Omega'} Z I(\lambda Z) \, \mathbb{P}(d\omega) = x_0.$$

Taking $Z = 1$ gives that X^* is constant and of course this constant must be x_0, which recovers the concave version of our zero-drift asset result, equation (1.5).

For general claims we can solve the $\tilde{F}_C$-problem giving a solution we shall denote $\tilde{X}^*$. Now in the $\tilde{F}_C$-problem for given ω' we wish to maximise

$$\tilde{F}_C(X(\omega')) - \lambda(Z(\omega')X(\omega') - x_0). \tag{1.7}$$

Where F_C and $\tilde{F}_C$ do not coincide (1.7) is linear because $\tilde{F}_C$ is linear. Over a linear portion we can move X either up or down so as to strictly not decrease (1.7). Hence we can modify our solution $\tilde{X}^*$ to give a solution X^* which is still optimal for $\tilde{F}_C$, and which only takes values for which F_C and $\tilde{F}_C$ coincide. Then for any X,

$$\mathbb{E}[F_C(X^*)] = \mathbb{E}[\tilde{F}_C(X^*)] \ge \mathbb{E}[\tilde{F}_C(X)] \ge \mathbb{E}[F_C(X)],$$

by definition of the concave relaxation. So in fact X^* is F_C-optimal. Again consider taking $Z = 1$. We have that X^* is a constant but if $\tilde{F}_C(x_0) \neq F_C(x_0)$ we must randomize X^* between constants where $\tilde{F}_C$ and F_C coincide. That is we recover the zero-drift case.

1.3 Discussion of Related Work

Problems of maximising the probability of meeting a claim revealed at horizon time have been considered in some recent works with a variety of restrictions being placed on the claim and on the wealth process used to meet it.

In Föllmer and Leukert [16] the tradeable asset is a general semimartingale, so $dX_t = \phi_t dP_t$ where P is a semimartingale. The claim is revealed at horizon time, so C is $\mathcal{F}_T$-measurable. Hence we wish to find

$$\sup_{\phi} \mathbb{P}\left(x_0 + \int_0^T \phi_t\, dP_t \geq C\right). \tag{1.8}$$

Both complete and incomplete markets are considered.

Considering the complete market first, let $A^* \in \mathcal{F}_T$ maximise $\mathbb{P}(A)$ subject to $\tilde{\mathbb{E}}[C\mathbb{1}_A] \leq x_0$, where $\tilde{\mathbb{P}}$ is the pricing probability measure, that is A^* is the largest set that we can afford to meet the claim on. Let ϕ^* be a perfect hedge for the knockout option $\tilde{C} = C\mathbb{1}_{A^*}$. They show that ϕ^* solves the original problem, that is we are reduced to finding the maximal success set.

The Neyman-Pearson Lemma is used to find the maximal success set. Define $\tilde{\mathbb{Q}}$ via

$$\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}} = \frac{C}{c_0},$$

where $c_0 = \tilde{\mathbb{E}}[C]$. So the constraint $\tilde{\mathbb{E}}[C\mathbb{1}_A] \leq x_0$ becomes $\tilde{\mathbb{Q}}(A) \leq x_0/c_0$, so x_0/c_0 will be the power of our test.

By the Neyman-Pearson Lemma

$$A^* = \left\{\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} > a^* C\right\},$$

where

$$a^* = \inf\left\{a : \tilde{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} > aC\right) \leq \frac{x_0}{c_0}\right\}.$$

Now $\tilde{\mathbb{Q}}(A) = x_0/c_0$ is satisfied if and only if $\mathbb{P}\left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = a^* C\right) = 0$. However, it may be that $d\mathbb{P}/d\tilde{\mathbb{P}} = a^* C$ with positive probability. This is addressed by randomizing but a slightly different problem must be considered.

It can be shown that if $\tilde{\phi}$ is the hedge for

$$\tilde{C} = C\left(\mathbb{1}_{\{\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} > a^* C\}} + \gamma \mathbb{1}_{\{\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = a^* C\}}\right),$$

where

$$\gamma = \frac{\frac{x_0}{c_0} - \tilde{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} > a^* C\right)}{\tilde{\mathbb{Q}}\left(\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} = a^* C\right)},$$

then $\tilde{\phi}$ maximises $\mathbb{E}[X_T/C \wedge 1]$. In Section 2.3 we shall show that optimising the expected wealth ratio is equivalent to optimising the probability of meeting the claim.

Consider the case of an incomplete market and again consider maximising the expected success ratio. We no longer have a unique equivalent martingale measure $\tilde{\mathbb{P}}$. However, providing

$$c_0 = \sup_{\tilde{\mathbb{P}}} \tilde{\mathbb{E}}[C],$$

is finite, we may take $\tilde{\phi}$ to be the superhedging strategy from decomposing

$$\tilde{C}_t = \sup_{\tilde{\mathbb{P}}} \tilde{\mathbb{E}}[\tilde{C}|\mathcal{F}_t],$$

as

$$\tilde{C}_t = c_0 + \int_0^t \tilde{\phi}_s \, dP_s - D_t$$

where D is an increasing optional, but not necessarily previsible, process. This corresponds to the cumulative withdrawal of wealth allowed by the superhedging strategy as one learns about the development of the underlying price process P. Decompositions of this kind are discussed in El Karoui and Quenez [34]. It can be shown that if we now take $\tilde{\phi}$ as the strategy for trading in the the tradeable asset this maximises the expected wealth ratio.

However, the condition that

$$c_0 = \sup_{\tilde{\mathbb{P}}} \tilde{\mathbb{E}}[C] < \infty,$$

is noticeably restrictive. For example it excludes the imperfectly correlated geometric-Brownian model that we shall look at in Chapter 2.

Karatzas [32] takes a similar approach, with a similarly restrictive condition, but has parameter uncertainty.

In Spivak and Cvitanić [49] the tradeable asset is geometric-Brownian and the claim is revealed at horizon time. So

$$dX_t = \phi_t \sigma_t dW_t, \tag{1.9}$$

with W_t being a standard Brownian motion, and C is revealed at time T.

A duality approach is used. For $\zeta > 0$ take

$$\tilde{U}(\zeta) = \sup_{z \leq C} \left\{ \mathbb{1}_{\{z \leq 0\}} + \zeta z \right\} = \begin{cases} \zeta C & \text{if} \quad \zeta C \geq 1 \\ 1 & \text{if} \quad 0 \leq \zeta C < 1 \end{cases}. \tag{1.10}$$

Hence

$$\mathbb{1}_{\{C - X_T^\phi \leq 0\}} \leq \tilde{U}(\zeta) - \zeta(C - X_T^\phi).$$

Taking expectations we have

$$\begin{aligned} \mathbb{P}(X_T^\phi \geq C) &\leq \mathbb{E}[\tilde{U}(\zeta)] - \zeta \mathbb{E}[C - X_T^\phi] \\ &\leq \mathbb{E}[\tilde{U}(\zeta)] - \zeta(c_0 - x_0) \\ &= F_0(\zeta), \end{aligned}$$

where

$$F_0(\zeta) = \mathbb{P}(C\zeta < 1) - \zeta \left(c_0 - x_0 - \mathbb{E}\left[C \mathbb{1}_{\{C\zeta > 1\}} \right] \right) = \mathbb{E}[\tilde{U}(\zeta)] - \zeta(c_0 - x_0).$$

The latter term is linear while the former is the expectation of a convex function and so is convex, attaining its minimum at

$$\tilde{\zeta} = \inf\{\zeta > 0 : \mathbb{E}\left[C \mathbb{1}_{\{C\zeta > 1\}} \right] \geq c_0 - x_0\}.$$

We have $\mathbb{P}(X_T^\phi \geq C) = F_0(\tilde{\zeta})$ for some admissible ϕ, i.e. that ϕ is optimal, if and only if

$$\mathbb{E}[X_T^\phi] = x_0,$$

and

$$C - X_T = C \left(\mathbb{1}_{\{\tilde{\zeta} C > 1\}} + \mathbb{1}_{E \cap \{\tilde{\zeta} C = 1\}} \right),$$

for some set E.

We randomize over the set E since we cannot afford to always meet the claim for $C = \frac{1}{\zeta}$ but if we never met it for such C we would have spare money left.

We note that the duality approach is essentially the same as the Lagrangian approach used for a general asset in a complete market in Section 1.1. The conclusion drawn is the same.

Spivak and Civtanić [49] actually have a slightly more general wealth process including interest rates

$$dX_t = r_t X_t dt + \phi_t \sigma_t dW_t,$$

but the use of discounted variables essentially removes the need for this. There is also a margin requirement, so a policy ϕ is only admissible if

$$X_t^{\phi} \geq A_t,$$

but this simply corresponds to a translation of the wealth process.

In Föllmer and Leukert [17] the related problem of minimizing the expected shortfall is considered, that is the problem of finding $\inf_{\phi} \mathbb{E}[(C_T - X_T^{\phi})^{+}]$ is considered. Pham [45] relaxes the conditions of Föllmer and Leukert [17]. This problem is also considered in Schulmerich and Trautmann [47] but there a linear programming approach is taken.

Chapter 2

Claims Revealed Continuously Through Time

Now we have our claim being revealed continuously through time. This is more realistic for practical applications like our examples of a trader trying to meet his targets or a company hedging against a claim on a restricted or illiquid stock. It also allows us to specify some correlation between the claim and the trading wealth, which itself adds greater realism. However, in specifying how the claim is revealed we make our model much more specific.

2.1 Discussion of Related Work

In one of the earliest articles in this area, Ferguson [15] minimizes the probability of ruin over an infinite horizon with discrete time and a discrete wealth space.

During the 1980s there was considerable interest in problems of controlling diffusions to a goal, such as Heath et al [21]. Pestien and Sudderth [43] maximise the probability of survival over an infinite horizon with Brownian wealth by controlling the diffusion coefficient. Pestien and Sudderth [44] do the same but there they control the diffusion coefficient and the drift. Such survival probabilities are also considered by Majumder and Radner [37] who take wealth to behave as a limit of Markov chains.

In more recent years there has been some work on problems of meeting a claim through continuous time to a fixed horizon. These have all taken the claim to be constant.

Kulldorff [36] has a geometric-Brownian wealth process

$$\frac{dX_t}{X_t} = \sigma_t(\mu_t dt + dW_t),$$

with σ_t being the control variable and μ_t being a given deterministic function. The claim is constant and without loss of generality $C = 1$.

First consider the problem with value function

$$V(x,t) = \sup_{\sigma} \mathbb{E}[X_T \wedge 1 \mid X_t = x],$$

i.e. consider the payoff $V(x,T) = x \wedge 1$. The state space is

$$S = \{(x,t) : 0 \leq x \leq 1,\ 0 \leq t \leq T\}.$$

Let τ be the first time that X_t hits the boundary of S. By Itô's Lemma

$$\mathbb{E}[V(X_\tau,\tau)|X_t] = V(X_t,t) + \mathbb{E}\left[\int_t^\tau I(\sigma)\,ds \middle| X_t\right],$$

where

$$I(\sigma) = \dot{V}(X_s,s) + \sigma_s\mu_s V'(X_s,s) + \frac{1}{2}V''(X_s,s)\sigma_s^2.$$

So

$$\mathbb{E}[V(X_\tau,\tau)|X_t] \begin{cases} = V(X_t,t) & \text{if } I(\sigma) = 0 \text{ for all } t \leq s \leq \tau \\ \leq V(X_t,t) & \text{otherwise} \end{cases}.$$

Now

$$\sigma_\varepsilon^* = \Phi'(\Phi^{-1}(x))\frac{m_t}{\mu_t},$$

where Φ and Φ' are the normal cumulative distribution and density functions respectively and $m_t = \sqrt{\int_t^T (\mu_s^+)^2\,ds}$, is a solution to $I(\sigma) = 0$ and so is an optimal control. The value function will satisfy $I(\sigma) = 0$. The function

$$V_\varepsilon(u,t) = \Phi(\Phi^{-1}(x) + m_t),$$

satisfies this and so is the value function. We shall consider existence and uniqueness for partial differential equations of this form in Section 2.2.

However, we are more concerned with the payoff $V(x,T) = \mathbb{1}_{\{x\leq 1\}}$. Here Kulldorff [36] uses a martingale argument to show that the two problems are equivalent. We use similar arguments in a broader context in Section 2.3.

Heath [20] considers the same problem as Kulldorff [36]. However, he uses the Neyman-Pearson Lemma to derive his test value function.

Assume $\mu_t = \mu$, a constant. Take

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathcal{F}_T} = e^{\mu W_T + \frac{1}{2}\mu^2 T} = e^{\mu W_T^{\mathbb{Q}} - \frac{1}{2}\mu^2 T},$$

where, under $\mathbb{Q}$, $dX_t = \sigma_t dW_t^{\mathbb{Q}}$. Suppose we can find a unique λ such that

$$\mathbb{Q}\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \geq \lambda\right) = x_0,$$

then

$$\mathbb{P}\left(\frac{d\mathbb{P}}{d\mathbb{Q}} \geq \lambda\right) = \sup\{\mathbb{P}(B) : \mathbb{Q}(B) \leq x_0\},$$

by the Neyman-Pearson Lemma.

Now

$$\mathbb{Q}\left(e^{\mu W_T^{\mathbb{Q}} - \frac{1}{2}\mu^2 T} \geq \lambda\right) = x_0$$

gives

$$\log \lambda = -\mu\sqrt{T}\Phi^{-1}(x_0) - \frac{1}{2}\mu^2 T,$$

and

$$\mathbb{P}\left(\mu W_T + \frac{1}{2}\mu^2 T \geq -\mu\sqrt{T}\Phi^{-1}(x_0) - \frac{1}{2}\mu^2 T\right) = \Phi\left(\Phi^{-1}(x_0) + \mu\sqrt{T}\right).$$

We can show directly that this satisfies the partial differential equation of Kulldorff [36].

Browne [6] has n assets with geometric-Brownian prices

$$\frac{dP_t^i}{P_t^i} = \mu_t^i dt + \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j.$$

There is also a riskless asset paying a deterministic interest rate but we can ignore this by considering discounted prices. Consequently the wealth process is

$$dX_t^\phi = \sum_{i=1}^{n} \phi_t^i \frac{dP_t^i}{P_t^i}.$$

The ϕ_t^i give the amount of money invested in asset i. The claim is constant $C = c$. In Section 3.1 we consider a system of multiple traded assets with a claim revealed continuously through time.

Itô's Lemma and control theory arguments give us that the value function satisfies

$$\sup_{\phi}\{\frac{1}{2}V_{xx}\phi_t^T\sigma_t\sigma_t^T\phi_t + V_x\phi_t^T\mu_t\} + \dot{V} = 0,$$

so the optimal policy is

$$\phi_t^* = -\sigma^{-1\,T}\sigma_t\mu_t\frac{V_x}{V_{xx}},$$

giving that the value function satisfies

$$\xi_t\frac{V_x^2}{2V_{xx}} - \dot{V} = 0,$$

where $\xi_t = \mu_t^T\sigma_t^{-1\,T}\sigma_t^{-1}\mu_t$. We note that this is essentially a special case of the partial differential equation we obtain in Section 2.2. Again the function

$$V(x,t) = \Phi\left(\Phi^{-1}\left(\frac{x}{c}\right) + \sqrt{\int_t^T \xi_s ds}\right)$$

satisfies this and so is the value function.

Browne also considers a number of related problems: maximising the probability of reaching a constant claim before ruin and within a time horizon [4], maximising the probability of meeting a constant claim given an enforced continuous withdrawal of funds at fixed rate [5], a combination of optimising over a time goal and maximising success probability [7]. This last article only considers the complete market case.

The key difference between what we shall do in the sequel and the literature we have discussed in this section is that we shall have a random claim.

2.2 The Bellman Equation

We specify how the claim will be revealed and how our price process will evolve using geometric Brownian motions thus

$$\frac{dC_t}{C_t} = \sigma_t dW_t, \quad C_0 = c_0, \tag{2.1}$$

$$\frac{dP_t}{P_t} = \rho_t dW_t + \sqrt{1-\rho_t^2}dB_t + \mu_t dt, \tag{2.2}$$

where W_t and B_t are independent Brownian motions. Compared with the formulation we gave in the introduction, (1) and (2), we have omitted the contingent claim drift ν and the multiplier term η in the asset price process. The former is explained in Section

3.1 and the latter is explained by a simple redefinition of the price process.

The parameter ρ gives the correlation between the contingent claim and the tradeable asset. In particular, the case $\rho = 1$ corresponds exactly to a complete market. A strategy ϕ_t chosen by the agent controls how the wealth evolves,

$$\frac{dX_t^\phi}{X_t^\phi} = \phi_t \frac{dP_t}{P_t}, \quad X_0 = x_0 \tag{2.3}$$

That is ϕ_t is the proportion of one's wealth invested in the risky asset. Of course, one of the key reasons for choosing a geometric-Brownian model is it being both tractable and a reasonable model for the price of a financial security.

It will be convenient to reduce the number of variables in our problem by considering the ratio $U_t = X_t^\phi / C_t$. Then, by Itô's formula,

$$\frac{dU_t}{U_t} = dW_t \left(\phi_t \rho_t - \sigma_t\right) + dB_t \phi_t \sqrt{1-\rho_t^2} + dt \left(\phi_t \mu_t + \sigma_t^2 - \sigma_t \phi_t \rho_t\right), \tag{2.4}$$

that is U is an autonomous diffusion.

We can define the value function in terms of u alone,

$$V(u,t) = \sup_{(\phi_s):t \leq s \leq T} \mathbb{E}\left[\, \mathbb{1}_{\{U_T \geq 1\}} \middle| \, U_t = u \right].$$

The value function is a supermartingale under any policy ϕ and a martingale under the optimal policy ϕ^*. Hence,

$$V(u,t) = \sup_\phi \mathbb{E}[V(u_{t+dt}, t+dt) \,|\mathcal{F}_t].$$

Now applying Itô's formula we obtain the Hamilton-Jacobi-Bellman (HJB) equation for this problem thus

$$V(u,t) = \sup_\phi \mathbb{E}\left[V(u,t) + V_u dU_t + \frac{1}{2} V_{uu} (dU_t)^2 + \dot{V} dt\right].$$

So

$$0 = \sup_\phi \left\{ \frac{1}{2} V_{uu} u^2 \phi^2 + \phi(V_u u(\mu - \sigma\rho) - V_{uu} u^2 \sigma\rho) \right\} + \sigma^2 (V_u u + \frac{1}{2} V_{uu} u^2) + \dot{V}. \tag{2.5}$$

If $V_{uu} \leq 0$ then we have a unique finite maximum at

$$\phi^* = \frac{V_u(\mu - \sigma\rho)}{V_{uu} u} - \sigma\rho, \tag{2.6}$$

giving

$$\sigma^2(V_u u + \frac{1}{2}V_{uu}u^2) - \frac{\{V_u(\mu - \sigma\rho) - V_{uu}u\sigma\rho\}^2}{2V_{uu}} + \dot{V} = 0. \tag{2.7}$$

We have boundary conditions $V(0,t) = 0$, meaning that if we have no money we will definitely lose, and $V(u,t) \to 1$ as $u \to \infty$, meaning that as we have more and more money or a smaller and smaller claim we become increasingly certain of winning. We note that we have not given a rigorous argument in deriving this equation but for our purposes this heuristic explanation of why the equation is plausible will be entirely sufficient.

Consider making the transformation $Y(w,t) = V(e^w, t)$, then the partial differential equation,

$$\frac{1}{2}V_{uu}u^2\left((\phi\rho - \sigma)^2 + \phi^2(1-\rho^2)\right) + V_u u(\mu\phi - \sigma\rho\phi + \sigma^2) + \dot{V} = 0, \tag{2.8}$$

becomes

$$\begin{aligned}&\frac{1}{2}Y_{ww}\left((\phi\rho - \sigma)^2 + \phi^2(1-\rho^2)\right)\\ &\quad + Y_w\left(\mu\phi + \sigma^2 - \sigma\phi\rho - \frac{1}{2}(\phi\rho - \sigma)^2 - \frac{1}{2}\phi^2(1-\rho^2)\right) + \dot{Y} = 0,\end{aligned}$$

which is uniformly parabolic, since $(\phi\rho - \sigma)^2 + \phi^2(1-\rho^2) > 0$ for all w, t. Consequently the maximum principle applies, see for example Evans [13] p375, that is the maximum of the solution lies on the boundary.

Suppose the equation had two solutions Y^1 and Y^2, then $Y^1 - Y^2$ satisfies the equation also but with zero boundary conditions so

$$\max_w(Y^1 - Y^2) = 0,$$

that is $Y^1 - Y^2 \leq 0$. Interchanging the role of Y^1 and Y^2 we also have $Y^1 - Y^2 \geq 0$ so in fact $Y^1 = Y^2$, i.e. the solution is unique. That is we have a unique solution to (2.8) for any strategy $\phi(u,t)$. So we have a unique solution for the optimal strategy $\phi^*(u,t)$. The optimal strategy is unique. We conclude that we have a unique solution to (2.5).

Further, the partial differential equation theory tells us that a parabolic equation with continuous coefficients has at least one solution over any given bounded time interval, see for example Friedman [19] p241.

2.3 Payoff Equivalences

We are interested in solving the problem $\sup_\phi \mathbb{P}\left(X_T^\phi \geq C_T\right)$ which corresponds to having a payoff function of $V(u,T) = \mathbb{1}_{\{u\geq 1\}}$. However, we will now show that, in our setting, for $t < T$, we obtain the same value function if we use the boundary condition $V(u,t) = u \wedge 1$.

The payoff $V(u,T) = u \wedge 1$ is continuous and concave whereas our original payoff is not. It is found necessary in Föllmer and Leukert [16] to consider this payoff. There it is justified by heuristic comparison with the Neyman-Pearson theory. We will see in Chapter 3 that it is sufficient to consider the zero correlation case.

Theorem 2.1 *We have the following equivalence of payoffs for our model with dynamics given by (2.1)-(2.2) in the case of zero correlation, for $t < T$:*

$$\operatorname*{ess\,sup}_{\phi_s : t\leq s\leq T} \mathbb{P}\left(U_T^\phi \geq 1 \middle| \mathcal{F}_t\right) = \operatorname*{ess\,sup}_{\phi_s : t\leq s\leq T} \mathbb{E}\left[U_T^\phi \wedge 1 \middle| \mathcal{F}_t\right].$$

Proof. First suppose $x_0 \leq c_0$ and suppose temporarily that the time horizon is very short. Choose, for $0 \leq t \leq \varepsilon$, the policy

$$\theta_t = \frac{\operatorname{sgn}(\mu_t)}{X_t\sqrt{\varepsilon - t}} \mathbb{1}_{\{0<X_t<c_0\}},$$

where we adopt the convention that $\operatorname{sgn}(0) = 1$. This gives the wealth process

$$dX_t^\theta = \left\{\frac{\operatorname{sgn}(\mu_t)}{\sqrt{\varepsilon - t}} dB_t + \frac{|\mu_t|}{\sqrt{\varepsilon - t}} dt\right\} \mathbb{1}_{\{0<X_t<c_0\}}.$$

Suppose we take

$$dY_t = \frac{\operatorname{sgn}(\mu_t)}{\sqrt{\varepsilon - t}} dB_t \mathbb{1}_{\{0<Y_t<c_0\}}, \quad Y_0 = x_0.$$

Up to the time when the first one exits $(0, c_0)$, X^θ and Y have the same volatility and so we conclude that $X^\theta \geq Y$.

Hence

$$\sup_\phi \mathbb{P}\left(X_\varepsilon^\phi \geq c_0\right) \geq \mathbb{P}\left(X_\varepsilon^\theta \geq c_0\right) \geq \mathbb{P}\left(Y_\varepsilon \geq c_0\right).$$

We can think of Y as a time change of a Brownian motion,

$$Y_t = \beta_{A_t},$$

where

$$A_t = \int_0^t \frac{\mathbb{1}_{\{0<Y_s<c_0\}}}{\varepsilon - s} ds.$$

Now β_t almost surely hits 0 or c_0 eventually and $A_t \uparrow \infty$ as $t \uparrow \varepsilon$ if Y_t does not exit $(0, c_0)$. So Y_t is almost sure to exit $(0, c_0)$ by time ε. This will allow us to apply the Optional Stopping Theorem.

Let $H_a = \inf\{t \geq 0 : Y_t = a\}$, i.e. the first time Y_t hits the level a. Now

$$\mathbb{P}(Y_\varepsilon \geq c_0) = \mathbb{P}(H_{c_0} < H_0) = \frac{1}{c_0}\mathbb{E}[Y_{H_{c_0} \wedge H_0}] = \frac{x_0}{c_0}.$$

On the other hand if $x_0 > c_0$ then taking $\theta = 0$,

$$\mathbb{P}\left(X_\varepsilon^\theta \geq c_0\right) = 1.$$

Combining the two cases

$$\sup_\phi \mathbb{P}(X_\varepsilon^\phi \geq c_0) \geq \frac{x_0}{c_0} \wedge 1.$$

Our intuition for this bound is that as the time horizon is short, we are unable to take advantage of the drift. Instead we gamble on reaching c_0, with martingale odds.

We are in fact not concerned with $\mathbb{P}\left(X_\varepsilon^\phi \geq c_0\right)$ but rather with $\mathbb{P}\left(X_\varepsilon^\phi \geq C_\varepsilon\right)$. Consider the chance of the claim moving a relatively long way in a short time,

$$p(\varepsilon) = \mathbb{P}\left(C_\varepsilon > c_0\left(1 + \varepsilon^{\frac{1}{4}}\right)\right) = 1 - \Phi\left(\frac{\log(1 + \varepsilon^{\frac{1}{4}}) + \frac{1}{2}\int_0^\varepsilon \sigma_t^2\, dt}{\sqrt{\int_0^\varepsilon \sigma_t^2\, dt}}\right),$$

where Φ is the normal distribution function. Note that $p(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. It follows that, for general $x_0 \geq 0$,

$$\sup_\phi \mathbb{P}(X_\varepsilon \geq C_\varepsilon) \geq \sup_\phi \mathbb{P}\left(X_\varepsilon \geq c_0\left(1 + \varepsilon^{\frac{1}{4}}\right), C_\varepsilon \leq c_0\left(1 + \varepsilon^{\frac{1}{4}}\right)\right) \geq \frac{1 - p(\varepsilon)}{1 + \varepsilon^{\frac{1}{4}}}\left(\frac{x_0}{c_0} \wedge 1\right). \tag{2.9}$$

Now we wish to extend this result to arbitrary time horizons. Suppose the strategy π is such that $(\pi_t)_{0 \leq t \leq T-\varepsilon}$ achieves within ε of the maximal value for $\mathbb{E}[U_{T-\varepsilon} \wedge 1]$ and $(\pi_t)_{T-\varepsilon < t \leq T}$ achieves within ε of the maximal value for $\mathbb{P}(X_T^\theta \geq C_T | \mathcal{F}_{T-\varepsilon})$, then

$$\mathbb{P}(X_T^\pi \geq C_T) = \mathbb{E}[\mathbb{P}(X_T^\pi \geq C_T | \mathcal{F}_{T-\varepsilon})] \geq \left(\frac{1 - p(\varepsilon)}{1 + \varepsilon^{\frac{1}{4}}}\right)\left(\mathbb{E}[U_{T-\varepsilon}^\pi \wedge 1] - \varepsilon\right),$$

using (2.9). So

$$\begin{aligned}\sup_{\phi}\mathbb{P}(X_T^{\phi}\geq C_T) &\geq \liminf_{\varepsilon\downarrow 0}\sup_{\phi}\left(\frac{1-p(\varepsilon)}{1+\varepsilon^{\frac{1}{4}}}\right)\left(\mathbb{E}[U_{T-\varepsilon}^{\phi}\wedge 1]-\varepsilon\right)\\ &\geq \sup_{\phi}\liminf_{\varepsilon\downarrow 0}\left(\frac{1-p(\varepsilon)}{1+\varepsilon^{\frac{1}{4}}}\right)\left(\mathbb{E}[U_{T-\varepsilon}^{\phi}\wedge 1]-\varepsilon\right).\end{aligned}$$

Using Fatou's Lemma,

$$\begin{aligned}\sup_{\phi}\mathbb{P}(X_T^{\phi}\geq C_T) &\geq \sup_{\phi}\mathbb{E}\left[\liminf_{\varepsilon\downarrow 0}(U_{T-\varepsilon}^{\phi}\wedge 1)\right]\\ &= \sup_{\phi}\mathbb{E}[U_T^{\phi}\wedge 1].\end{aligned}$$

However, we also have $\mathbb{1}_{\{U_T\geq 1\}}\leq U_T\wedge 1$ so the reverse inequality is trivial and

$$\sup_{\phi}\mathbb{P}(U_T^{\phi}\geq 1)=\sup_{\phi}\mathbb{E}[U_T^{\phi}\wedge 1].$$

□

We shall use this result in Section 2.4 to show that the value functions we derive do solve the problem we are concerned with. The proof of the result shows that the essential suprema are attained by strategies that are the same up to time T.

2.4 A Complete Market

In the complete market case we either know or are able to conjecture the optimal solution by comparison with the discrete-time case. We can then obtain the corresponding value function and show that it satisfies our partial differential equation. When we want to make explicit the dependence of the value function on the claim volatility σ, asset drift μ and correlation ρ, we will write $V(u,t;\sigma_{\bullet},\mu_{\bullet},\rho_{\bullet})$.

When the claim is constant, $C_t=c$, we are reduced to the case of the claim being revealed at time 0.

Proposition 2.1 *In the case of a constant claim and zero-drift asset the value function is given by*

$$V(u,t;0,0,0)=u\wedge 1.$$

Proof. We saw earlier (1.1) that it is optimal to choose

$$X_T^* = \begin{cases} c & \text{with probability } p \\ 0 & \text{with probabiltiy } 1-p \end{cases}, \tag{2.10}$$

with p chosen so that $\mathbb{E}[X] = x_0$. The required p is $p = \frac{x_0}{c} \wedge 1$. $\square$

We note that this satisfies our HJB equation (2.7).

Proposition 2.2 *For a constant claim but a tradeable asset with constant non-zero drift we have, for $t < T$, the value function*

$$V(u, t; 0, \mu, 0) = \Phi(|\mu|\sqrt{T-t} + \Phi^{-1}(u)). \tag{2.11}$$

Proof. Recall that $\tilde{\mathbb{P}}$ is the minimal martingale measure and that $Z = d\tilde{\mathbb{P}}/d\mathbb{P}$. Suppose we aim to obtain

$$X_T^* = c\mathbb{1}_{\{Z_T \leq z^*\}}, \tag{2.12}$$

where z^* is such that the budget constraint,

$$\mathbb{E}[Z_T X_T | \mathcal{F}_t] = x,$$

is satisfied. Now

$$Z_t = e^{-\mu\beta_t - \frac{1}{2}\mu^2 t},$$

where $d\beta_t = \rho dW_t + \sqrt{1-\rho^2}dB_t$ defines a standard Brownian motion. So, with $Z_t = z$,

$$\begin{aligned} \mathbb{P}(Z_T \leq y|\mathcal{F}_t) &= \mathbb{P}\left(\exp\left\{-\mu(\beta_T - \beta_t) - \frac{1}{2}\mu^2(T-t)\right\} \leq \frac{y}{z}\right) \\ &= \Phi\left(\frac{\log\frac{y}{z} + \frac{1}{2}\mu^2(T-t)}{|\mu|\sqrt{T-t}}\right), \end{aligned} \tag{2.13}$$

and the budget constraint,

$$\mathbb{E}\left[X_T \frac{Z_T}{Z_t}\middle|\mathcal{F}_t\right] = x,$$

can be written as

$$\frac{c}{z}\int_{-\infty}^{z^*} \frac{y}{|\mu|\sqrt{T-t}}\frac{1}{y\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\frac{\log\frac{y}{z} + \frac{1}{2}\mu^2(T-t)}{|\mu|\sqrt{T-t}}\right)^2\right\} dy = x,$$

which becomes

$$c\int_{-\infty}^{\log\left(\frac{z^*}{z}\right)} \frac{1}{|\mu|\sqrt{T-t}}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\left(\frac{s+\frac{1}{2}\mu^2(T-t)}{|\mu|\sqrt{T-t}}\right)^2\right\} ds = x,$$

on making the substitution

$$s = \log\frac{y}{z}.$$

This gives

$$\Phi\left(\frac{\log\left(\frac{z^*}{z}\right) - \frac{1}{2}\mu^2(T-t)}{|\mu|\sqrt{T-t}}\right) = \frac{x}{c}.$$

Hence we find, using (2.13), that for the target wealth given in (2.12)

$$\begin{aligned}\mathbb{P}(X_T \geq c|\mathcal{F}_t) &= \mathbb{P}(Z_T \leq z^*|\mathcal{F}_t)\\ &= \Phi(|\mu|\sqrt{T-t} + \Phi^{-1}(u)).\end{aligned}$$

With $\sigma = 0$ our HJB equation (2.7) reduces to

$$2\dot{V}V'' - (V'\mu)^2 = 0 \tag{2.14}$$

which the $\mathbb{P}(X_T \geq c|\,\mathcal{F}_t)$ obtained above satisfies. To show that this gives the value function it remains to show that it satisfies the boundary condition at $t = T$. Substituting $t = T$ into (2.11) gives

$$\Phi\left(\Phi^{-1}(u)\right) = u \wedge 1.$$

However, Theorem 2.1 shows that for any time $t < T$ the value function for the problem with terminal payoff $V(u,T) = u \wedge 1$ is the same as that for the problem with terminal payoff $V(u,T) = \mathbb{1}_{\{u\leq 1\}}$. Consequently we do in fact have the correct value function.

□

We note that taking $\mu = 0$ recovers our earlier zero-drift asset case (2.10). In the general case, we can obtain the optimal policy, for $t < T$, from the value function,

$$\begin{aligned}\phi^* &= \frac{V_u\mu}{V_{uu}u}\\ &= \frac{\Phi'(\Phi^{-1}(u))}{u\sqrt{T-t}}\end{aligned}$$

In the zero-drift asset case there was some flexibility over the policy, providing it gave the appropriate wealth distribution, this is not so here. Although the claim is constant, it varies with respect to the state price density. At time T the optimal policy is given by the proof of Theorem 2.1.

However, we do note that, surprisingly, the optimal policy is independent of μ. Browne [6] notes this also. We see that this is plausible as follows. We choose to obtain

$$Z_T X_T^* = cZ_T \mathbb{1}_{\{Z_T \leq z^*\}}.$$

Now cZ_T decreases for increasing μ. On the other hand the event $\{Z_T \leq z^*\}$ occurs more for increasing μ. These two effects cancel each other out to give a policy which is independent of μ.

Browne [6] also considers, for given time-to-go, for what wealth ratios one is prepared to use a policy where the holding in cash is negative. This *borrowing region* is given by

$$\Gamma(u,t) = \{u : \phi_t^*(u) \geq 1\},$$

and so in the constant claim case is bounded above by the solution to

$$\Phi'(\Phi^{-1}(u)) - u\sqrt{T-t} = 1.$$

A plot of this region is given in Figure 2-1. We see that the critical wealth ratio tends to 1 with decreasing time-to-go and to 0 with increasing time-to-go as we would expect. We also note that it is convex. That is as we get closer to the horizon time the rate of increase of our critical wealth ratio increases.

The case with non-zero claim volatility, but zero-drift asset is very similar.

Proposition 2.3 *For a non-constant claim but a zero-drift tradeable asset we have the value function*

$$V(u,t;\sigma,0,1) = \Phi(\sigma\sqrt{T-t} + \Phi^{-1}(u)).$$

Sketch Proof We conjecture that it is optimal to choose to obtain

$$X_T^* = C_T \mathbb{1}_{\{C_T \leq c^*\}},$$

where c^* is such that $\mathbb{E}[X_T^*] = x$, recalling that the pricing measure is the real measure since the asset is zero-drift. A calculation which is entirely analogous to that in Proposition 2.2 gives that the value function for the conjectured optimal wealth is

$$V(u,t) = \Phi(\sigma\sqrt{T-t} + \Phi^{-1}(u)),$$

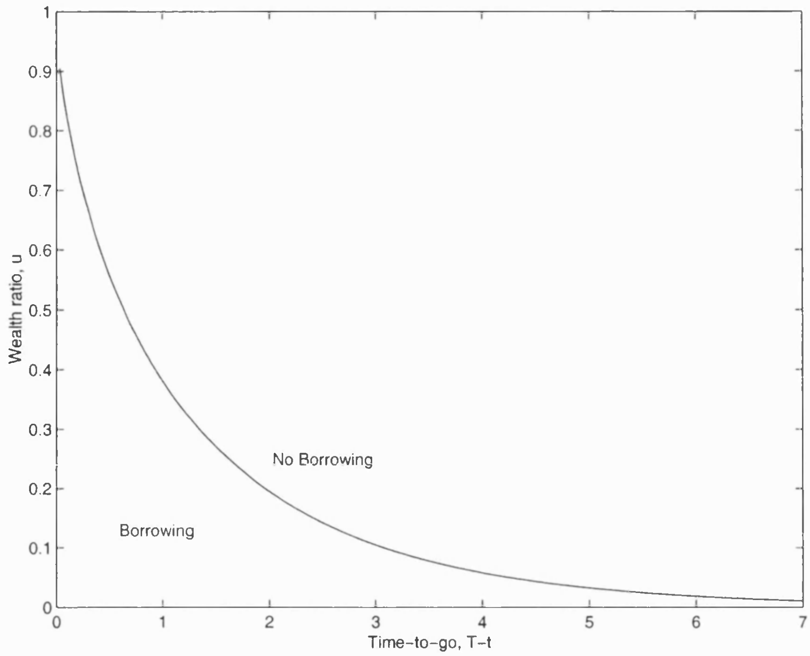

Figure 2-1: The Borrowing Region in the Continuous-time Constant-Claim Case

which, by analogy with the value function there, clearly solves the required equation of

$$\dot{V}V_{uu} - \frac{1}{2}\sigma^2 V_u^2 = 0.$$

So the conjectured optimal wealth is indeed optimal. □

We find that the optimal policy can be obtained as follows,

$$\begin{aligned}\phi^*(u) &= \frac{-V_u}{V_{uu}u} - \sigma \\ &= \frac{-\Phi'(\Phi^{-1}(u))}{u\sigma\sqrt{T-t}} - \sigma.\end{aligned}$$

We can again find the borrowing region. In this case it is bounded above by the solution to

$$\Phi'(\Phi^{-1}(u)) - \sigma(\sigma+1)u\sqrt{T-t} = 0.$$

This is plotted for $\sigma = 0.5$ and $\sigma = 1$ in Figures 2-2 and 2-3 respectively.

As we would expect for larger volatilities the wealth ratio for which we are prepared to borrow increases sharply for small time-to-go. We note that for small σ we are prepared to borrow for higher wealth ratios than in the general constant claim case.

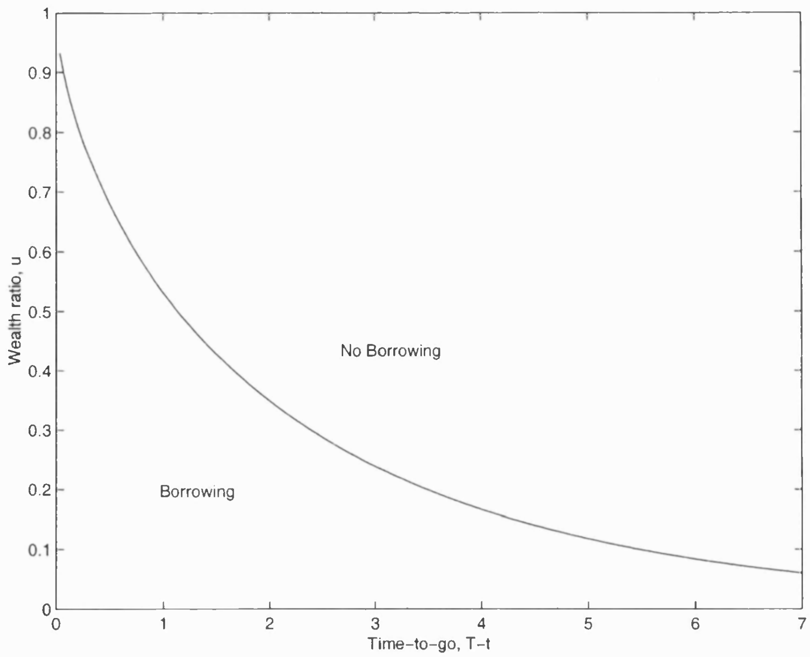

Figure 2-2: The Borrowing Region in the Complete-Market Martingale-Asset Case with claim volatility $\sigma = 0.5$

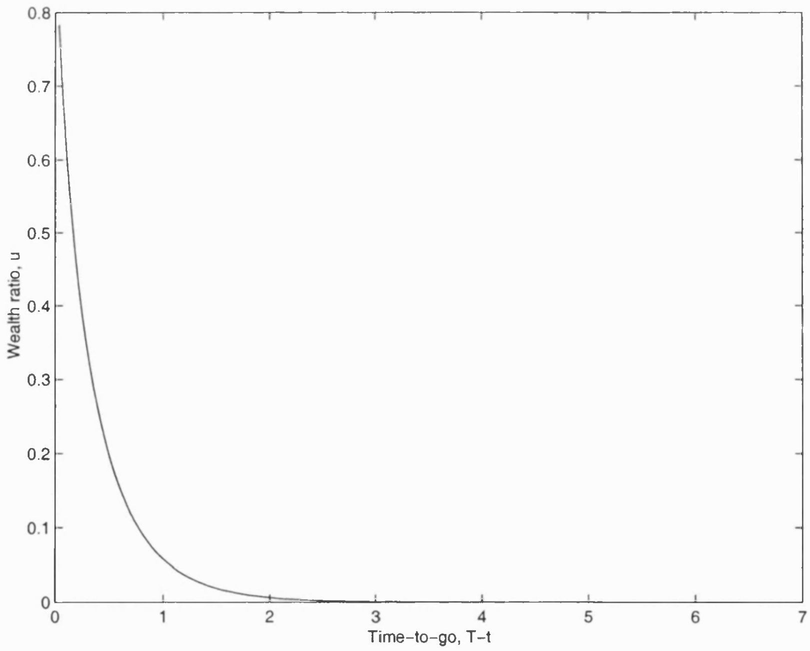

Figure 2-3: The Borrowing Region in the Complete-Market Maringale-Asset Case with claim volatility $\sigma = 1$

We obtain essentially identical partial differential equations since our problem is essentially the same in the two different cases but with the claim value replacing the state price density. The volatility σ of the claim corresponds to drift μ of the tradeable asset.

Proposition 2.4 *In the completely general complete market case we have value function*

$$V(u, t; \sigma, \mu, 1) = \Phi(\Phi^{-1}(ue^{\mu\sigma(T-t)}) + |\sigma - \mu|\sqrt{T-t}). \tag{2.15}$$

Sketch Proof. Essentially the same calculation as the proof of Proposition 2.3.

Alternative Proof. Take $n = 1$, $\Gamma = (0)$ and $\gamma = (1)$ in Theorem 3.1. We see that we reduce this case to the case where the claim has zero volatility (and so the correlation between claim and tradeable asset is irrelevant). $\square$

Figures 2-4 and 2-5 show the borrowing region for $\sigma = 1$; and $\mu = 0.01$ and $\mu = 0.5$ respectively. We see that there is little variation for varying μ.

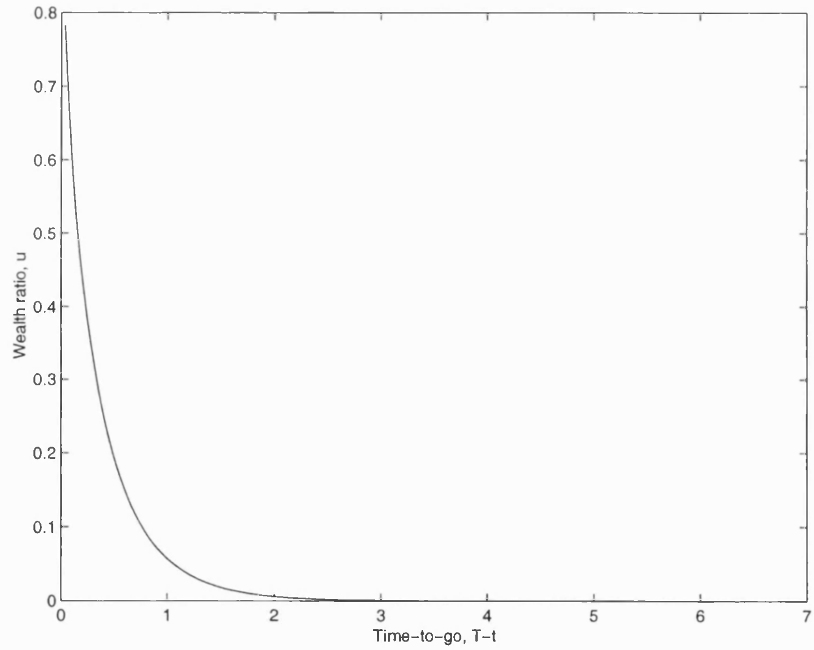

Figure 2-4: The Borrowing Region in the Complete-Market Case with Claim Volatility $\sigma = 1$ and Tradeable Asset Drift $\mu = 0.01$

We have again solved essentially the same problem with the value of the claim or the state price density being replaced by the claim adjusted by the state price density.

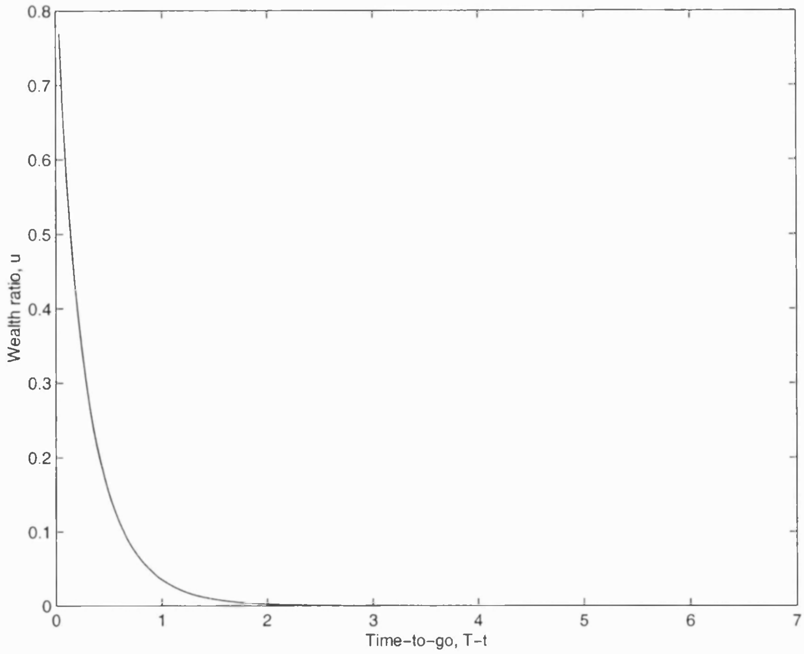

Figure 2-5: The Borrowing Region in the Complete-Market Case with Claim Volatility $\sigma = 1$ and Tradeable Asset Drift $\mu = 0.5$

The difference between claim volatility and asset drift $\sigma - \mu$ replaces drift μ or volatility σ.

A plot of the value function (2.15) is given in Figure 2-6. We note that $V(u,T) = u \wedge 1$ as we discussed earlier. We also note the flat region of the plot where success is almost guaranteed. This region is wider when there is more time to go.

Throughout this section we took μ and σ to be constant. We could however have worked with them as deterministic functions of time without making things practicably more complicated. The term $\sigma(T-t)$ would become $\sqrt{\int_t^T \sigma_s^2 \, ds}$. Other terms would change in a corresponding manner, for example, the value function (2.15) in the complete market case with non-zero drift would be

$$V(u,t) = \Phi\left(\Phi^{-1}\left(u \exp\left\{\int_t^T \mu_s \sigma_s \, ds\right\}\right) + \sqrt{\int_t^T (\mu_s - \sigma_s)^2}\right).$$

2.5 An Independent Claim and an Asset with Zero Drift

We will see in Corollary 3.1 that it is enough to focus on this case when the claim is independent of the tradeable asset. Here the continuous revealing of the claim makes more of a difference. When there is no drift in the asset that a trading strategy might

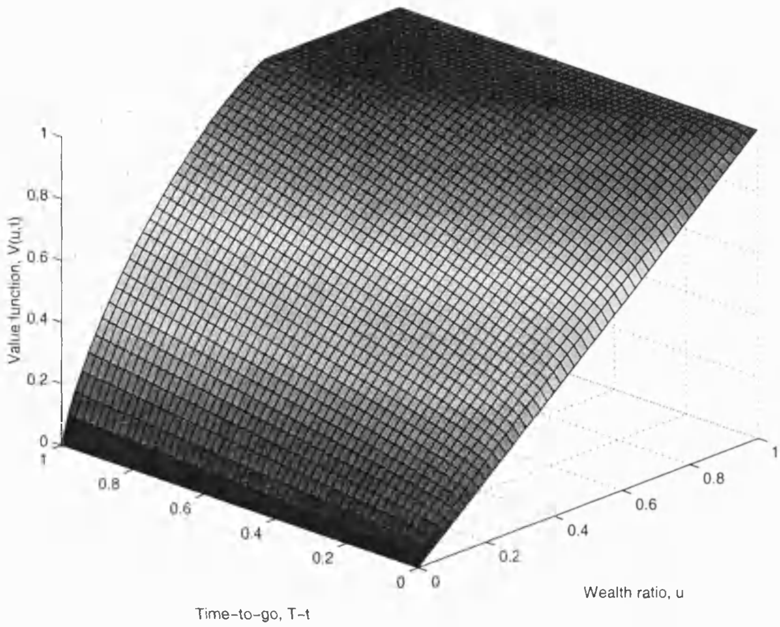

Figure 2-6: The Value Function in the Continuous-time Complete-Market case

take advantage of it seems best to put off any trading for as long as possible.

Proposition 2.5 *When the tradeable asset is zero-drift and the claim is independent of it, the value function is given by*

$$V(u,t) = \Phi\left(\frac{\log u + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) + \left(1 - \Phi\left(\frac{\log u + \frac{3}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\right) ue^{\sigma^2(T-t)}. \tag{2.16}$$

Proof. Suppose, as we suggested above, we defer all trading until just before T then, if $x_0 < C_T$, trade so as to have wealth C_T with probability $\frac{x_0}{C_T}$ and have nothing otherwise. This has

$$\begin{aligned}
\mathbb{P}(X_T \geq C_T | C_t) &= \int_0^\infty \mathbb{P}(C_T \in dy)\left(\frac{x_0}{y} \wedge 1\right) \\
&= \Phi\left(\frac{\log u + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) \\
&+ \int_{x_0}^\infty \frac{x_0}{y^2\sigma\sqrt{T-t}}\Phi'\left(\frac{\log\frac{y}{c} + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right) dy
\end{aligned}$$

which gives (2.16) on making the substitution $y = ce^r$. This can be confirmed to satisfy (2.7). Substituting $t = T$ gives $u \wedge 1$, and so, using Theorem 2.1, the result follows. □

A plot of (2.16) is given in Figure 2-7. We note that although there is a relatively flat region for large wealth ratio it is not as clearly demarcated as in the complete market case, (2.15). For the last small time interval before the horizon there is a sharp increase in the success probability for wealth ratio near 1 as it becomes clear that the chance of the claim escaping from reach in the time left is small.

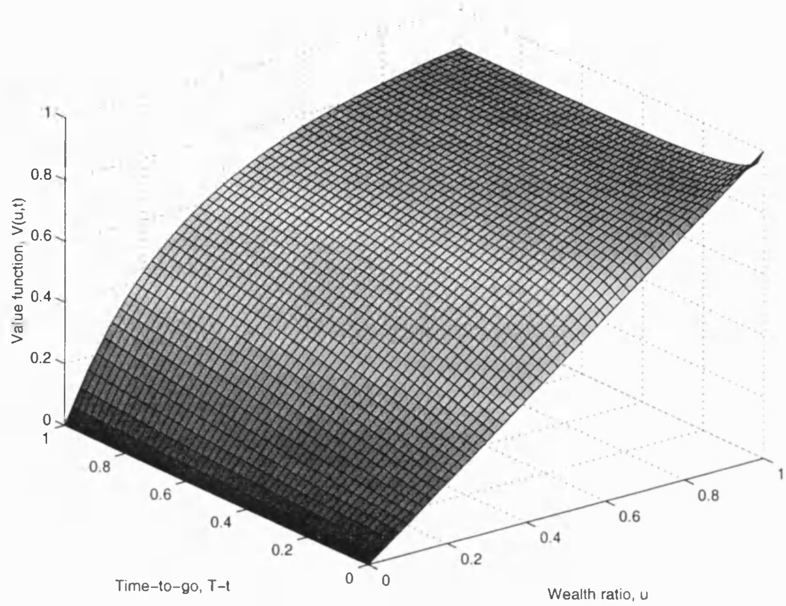

Figure 2-7: The Value Function in the Continuous-time Independent-Asset Zero-Drift Case

2.6 An Explicit Suboptimal Policy

In Section 2.5 we obtained an explicit formula for the value function of the problem with independent claim and zero-drift asset. We did this by conjecturing that the optimal policy would be to do nothing until just before the horizon time and then confirming that this is indeed optimal. In the case of an independent claim and tradeable asset with non-zero drift it is not clear what one would conjecture was the optimal policy. Consequently in Chapter 4 we shall be turning to numerical techniques for solving the HJB equation we have for the value function.

However, there is a policy which we can obtain as the solution of an algebraic equation which although not optimal we shall see later is quite effective. We shall see later that it is useful to have a good approximation to the optimal strategy. Suppose that for given wealth ratio u at time t we had to fix our policy at some constant value, and use that policy until horizon time, T. Choose the optimal such policy and call it $\hat{\phi}(u,t)$. Now consider using the dynamic policy $\hat{\phi}(U_s, s)$ of choosing at each instant the policy one would choose if one were never allowed to change policy in the future.

In Section 2.3 we saw that we should concern ourselves with optimizing $\mathbb{E}[U_T^\phi \wedge 1]$. Now, if the policy ϕ is constant

$$\log U_T \sim N\left(\log u + \left(\frac{1}{2}\sigma^2 + \phi\mu - \frac{1}{2}\phi^2\right)(T-t), (\phi^2+\sigma^2)(T-t)\right),$$

so, using Proposition 2.5,

$$\begin{aligned}\mathbb{E}[U_T \wedge 1] = \Phi&\left(\frac{\log u + \left(\frac{1}{2}\sigma^2 + \phi\mu - \frac{1}{2}\phi^2\right)(T-t)}{\sqrt{\phi^2+\sigma^2}\sqrt{T-t}}\right)\\ &+ ue^{(\sigma^2+\phi\mu)(T-t)}\left(1 - \Phi\left(\frac{\log u + \left(\frac{3}{2}\sigma^2 + \phi\mu + \frac{1}{2}\phi^2\right)(T-t)}{\sqrt{\phi^2+\sigma^2}\sqrt{T-t}}\right)\right).\end{aligned}$$

Differentiating and simplifying we find that $\hat{\phi}$ is given by

$$\begin{aligned}&\hat{\phi}\Phi'\left(\frac{\log u + \left(\frac{3}{2}\sigma^2 + \phi\mu + \frac{1}{2}\hat{\phi}^2\right)(T-t)}{\sqrt{\hat{\phi}^2+\sigma^2}\sqrt{T-t}}\right)\\ &\quad= \mu\sqrt{\hat{\phi}^2+\sigma^2}\sqrt{T-t}\left(1 - \Phi\left(\frac{\log u + \left(\frac{3}{2}\sigma^2 + \hat{\phi}\mu + \frac{1}{2}\hat{\phi}^2\right)(T-t)}{\sqrt{\hat{\phi}^2+\sigma^2}\sqrt{T-t}}\right)\right).\end{aligned} \tag{2.17}$$

We can solve this equation for $\hat{\phi}$ numerically, for example by a simple binary search. Figure 2-8 shows this optimal fixed policy $\hat{\phi}$ in the case with $\sigma = 1$ and $\mu = 0.1$. The largest stock holdings occur for small time-to-go, but there is some holding throughout time. For large time-to-go there is a much less marked increase in stock holding for decreasing wealth ratio.

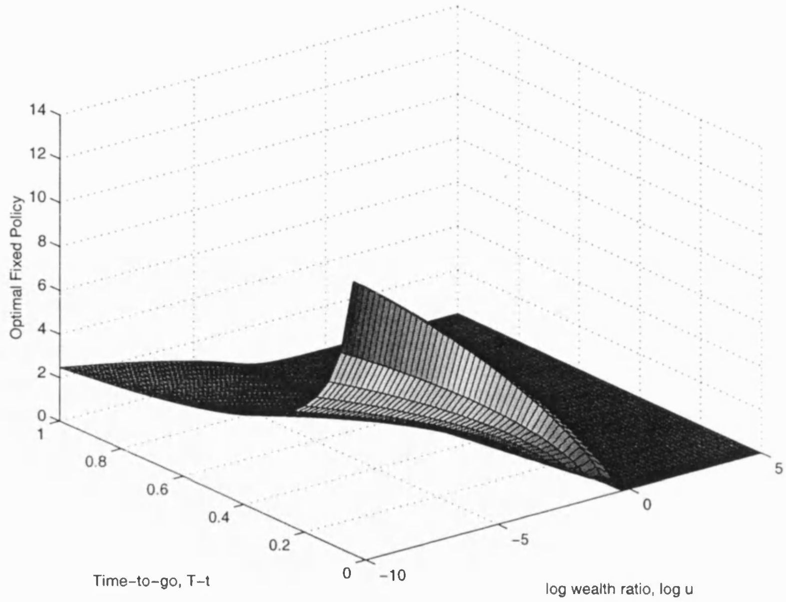

Figure 2-8: The Optimal Fixed Policy when $\sigma = 1$ and $\mu = 0.1$

Chapter 3

Related Problems

In this chapter we first consider a multiple asset generalisation of the problem we have so far considered and show that it reduces to the univariate case. In the second section we consider a variant of the quantile hedging problem with a different boundary condition.

3.1 Multiple Assets and Claims with Non-zero Drift

Suppose that instead of specifying how our claim was revealed and our price processes evolved using (2.1)–(2.2) we used the following model. The claim is as before except that it has a deterministic drift κ;

$$\frac{dC_t}{C_t} = \sigma_t dW_t + \kappa_t dt, \quad C_0 = c_0. \tag{3.1}$$

The single asset P is replaced by n assets indexed by i and solving the equations

$$\frac{dP_t^i}{P_t^i} = \eta_t^i \sum_{j=1}^{n} \Pi_t^{ij} dZ_t^j + \mu_t^i dt, \tag{3.2}$$

where the Brownian motions Z^j are independent. We suppose that the dependence between W and Z^j is given by the relationship

$$dW = \sum_{j=1}^{n} \rho_t^j dZ^j + \overline{\rho}_t d\overline{Z}, \tag{3.3}$$

where $\overline{Z}$ is a further Brownian motion independent of the Z^j, and again all parameters are deterministic. By assumption the market without the claim C is complete, and the number of traded assets equals the number of Brownian motions Z^j. Further, to exclude degeneracy and arbitrage we assume that Π_t is non-singular. If W is linearly dependent on these Z^j then the market remains complete with the introduction of C and $\overline{\rho} = 0$. Otherwise the introduction of the claim C makes the market incomplete.

In order to emphasise the similarities with previous sections we can rewrite (3.2) in the form

$$\frac{dP_t^i}{P_t^i} = \gamma_t^i dW_t + \sum_{j=1}^n \Gamma_t^{ij} dB_t^j + \mu_t^i dt, \tag{3.4}$$

where the Brownian motions B^j are a reparameterisation of the Z^j, but with the additional property that they are each independent of W.

We have

$$\begin{pmatrix} \Gamma & \gamma \\ 0 & \sigma \end{pmatrix} \begin{pmatrix} dB \\ dW \end{pmatrix} = \begin{pmatrix} D\Pi & 0 \\ \sigma\rho & \sigma\overline{\rho} \end{pmatrix} \begin{pmatrix} dZ \\ d\overline{Z} \end{pmatrix}, \tag{3.5}$$

where $D = D(\eta^1; \eta^2; \ldots; \eta^n)$, the diagonal matrix with entries η^1, η^2, $\ldots$, η^n. Rewriting

$$D\Pi dZ = \Gamma dB + \gamma dW,$$

gives

$$D\Pi\Pi^T D^T = \Gamma\Gamma^T + \gamma\gamma^T,$$

which we shall use in Theorem 3.1.

Given the formulation (3.1) - (3.3) we can define B and W by inverting on the left in (3.5). Conversely, given B and W we can define

$$dZ = \Pi^{-1}D^{-1}\Gamma dB + \Pi^{-1}D^{-1}\gamma dW,$$

and, if $\overline{\rho} \neq 0$,

$$d\overline{Z} = -\overline{\rho}^{-1}\rho\Pi^{-1}D^{-1}\Gamma dB + \overline{\rho}^{-1}dW.$$

If $\overline{\rho} = 0$, then $\overline{Z}$ is not necessary for the fomulation in (3.3).

The trading strategy of the agent is represented by the vector χ_t^i so that the dynamics of the trading wealth are given by

$$\frac{dX_t}{X_t} = \sum_{i=1}^n \chi_t^i \frac{dP_t^i}{P_t^i}, \quad X_0 = x_0. \tag{3.6}$$

Theorem 3.1 *The value function*

$$J_n(x/c, t_0; \sigma_\bullet, \kappa_\bullet, \mu_\bullet, [\gamma_\bullet, \Gamma_\bullet]) = \sup_{\chi_1, \ldots, \chi_n} \mathbb{P}(X_T \geq C_T | X_{t_0} = x, C_{t_0} = c)$$

of the n-dimensional problem, with price process dynamics given by (3.2), can be expressed in terms of the value function of our original one-dimensional problem with price dynamics (2.2),

$$J_n(u, t_0; \sigma_\bullet, \kappa_\bullet, \mu_\bullet, [\gamma_\bullet, \Gamma_\bullet]) = V\left(ue^{\int_{t_0}^T (\sigma_s \gamma_s^T G_s^{-1} \mu_s - \kappa_s)\, ds}, t_0; \tilde{\sigma}_\bullet, \tilde{\mu}_\bullet, 0\right),$$

where

$$\tilde{\sigma}_t^2 = \sigma_t^2 \left(1 - \gamma_t^T G_t^{-1} \gamma_t\right), \quad \tilde{\mu}_t = \left(\nu_t^T G_t^{-1} \nu_t\right)^{\frac{1}{2}},$$

and

$$G_t = \eta_t \Pi_t \Pi_t^T \eta_t^T = \Gamma_t \Gamma_t^T + \gamma_t \gamma_t^T, \quad \nu_t = \mu_t - \sigma_t \gamma_t.$$

There is a similar expression for the optimal policy,

$$\begin{aligned}
&\chi_t^*(u, t; \sigma_\bullet, \kappa_\bullet, \mu_\bullet, [\gamma_\bullet, \Gamma_\bullet]) \\
&= \left(\frac{\phi^*(ue^{\int_t^T (\sigma_s \gamma_s^T G_s^{-1} \mu_s - \kappa_s)\, ds}, t; \tilde{\sigma}_\bullet, \tilde{\mu}_\bullet, 0)}{\left(\nu_t G_t^{-1} \nu_t\right)^{\frac{1}{2}}}\right) G_t^{-1} \nu_t + \sigma_t G_t^{-1} \gamma_t,
\end{aligned}$$

where ϕ^* is the optimal policy in our original one-dimensional problem.

By assumption the traded assets are linearly independent so that $G_t = D_t \Pi_t \Pi_t^T D_t^T$ is non-singular and G_t is invertible as the theorem requires.

To prove the theorem we use the following lemma which follows directly from the Cauchy-Schwarz Inequality.

Lemma 3.1 *For $x, y \in \mathbb{R}^d$ and Λ a positive definite symmetric $d \times d$ matrix we have*

$$x^T y \le \left(x^T \Lambda x\right)^{\frac{1}{2}} \left(y^T \Lambda^{-1} y\right)^{\frac{1}{2}},$$

with equality when $x = c\Lambda^{-1} y$ for some $c > 0$.

Proof of Theorem 3.1. We prove the theorem when the current time is zero. Applying Itô's Formula to $\tilde{U} = X/C$, then analogously to (2.4),

$$\frac{d\tilde{U}_t}{\tilde{U}_t} = \sqrt{(\chi_t^T \gamma_t - \sigma_t)^2 + \chi_t^T \Gamma_t \Gamma_t^T \chi_t} d\beta_t + (\chi_t^T(\mu_t - \sigma_t \gamma_t) + \sigma_t^2 - \kappa_t) dt, \tag{3.7}$$

for β a Brownian motion.

Now if we take $\tilde{\chi}_t = \chi_t - \sigma_t G_t^{-1} \gamma_t$ some algebraic manipulation gives that

$$\tilde{\chi}_t^T G_t \tilde{\chi}_t + \sigma_t^2 (1 - \gamma_t^T G_t^{-1} \gamma_t) = \chi_t^T \Gamma_t \Gamma_t^T \chi_t + (\chi_t^T \gamma_t - \sigma_t)^2,$$

and so (3.7) becomes

$$\frac{d\tilde{U}_t}{\tilde{U}_t} = \sqrt{\tilde{\sigma}_t^2 + \tilde{\chi}_t^T G_t \tilde{\chi}_t}\, d\beta_t + (\tilde{\chi}_t^T \nu_t + \tilde{h}_t)dt,$$

where

$$\tilde{h}_t = \tilde{\sigma}_t^2 + \sigma_t \gamma_t^T G_t^{-1} \mu_t - \kappa_t.$$

Now define ϕ by $\phi_t^2 = \tilde{\chi}_t^T G_t \tilde{\chi}_t$ and $U = U^\phi$ via $U_0 = x/c$ and

$$\frac{dU_t^\phi}{U_t^\phi} = \sqrt{\tilde{\sigma}_t^2 + \phi_t^2}\, d\beta_t + (\phi_t \tilde{\mu}_t + \tilde{h}_t)dt.$$

We have $\tilde{\chi}_t^T \nu_t \le \phi_t \tilde{\mu}_t$ using the first part of Lemma 3.1, with $x = \tilde{\chi}_t$, $y = \nu_t$ and $\Lambda = G_t$. Consequently $\tilde{U}^\chi \le U^\phi$ using a stochastic comparison theorem. Further if we take

$$\tilde{\chi}_t^* = \left(\frac{\phi_t^*}{(\nu_t^T G_t^{-1} \nu_t)^{\frac{1}{2}}} \right) G_t^{-1} \nu_t,$$

then, using the second part of Lemma 3.1, $\tilde{\chi}_t^*$ is optimal. Hence

$$J_n(x/c, 0; \sigma_\bullet, \kappa_\bullet, \mu_\bullet, [\gamma_\bullet, \Gamma_\bullet]) = \sup_\phi \mathbb{E}[U_T^\phi \ge 1 | X_0/C_0 = x/c]$$

If we define

$$\hat{U}_t = U_t^\phi e^{\int_t^T (\sigma_s \gamma_s^T G_s^{-1} \mu_s - \kappa_s)\, ds},$$

then we find

$$\frac{d\hat{U}_t}{\hat{U}_t} = \sqrt{\tilde{\sigma}_t^2 + \phi_t^2} d\beta_t + \left(\phi_t \tilde{\mu}_t + \tilde{\sigma}_t^2\right) dt.$$

This is the same as (2.4), but with $\rho = 0$. Hence

$$\sup_\phi \mathbb{E}[U_T^\phi \ge 1 | X_0/C_0 = x/c] = V\left(u e^{\int_{t_0}^T (\sigma_s \gamma_s^T G_s^{-1} \mu_s - \kappa_s) ds}, 0; \tilde{\sigma}_\bullet, \tilde{\mu}_\bullet, 0 \right).$$

We can also read off the optimal strategy in the same fashion. □

This result is useful even in the one-dimensional case. There are many situations where a hedging instrument is neither the one that the contingent claim concerned is based on nor one that is entirely independent. For example, Hull [27], p37, describes a problem where an airline hedges its exposure to fluctuations in the price of jet fuel through trading in futures on domestic heating oil. It is also highly relevant to the problem of hedging a claim based on a basket of assets.

The introduction of correlation was one of the reasons for considering the problem with a continuously revealed claim.

Corollary 3.1 *The value function and optimal policy in the imperfectly correlated case can be expressed in terms of those for the case where the claim is fully independent of the tradeable asset,*

$$V(u,t_0;\sigma_\bullet,\mu_\bullet,\rho_\bullet) = V(ue^{\int_{t_0}^T \sigma_s\mu_s\rho_s\,ds}, t_0; \sigma_\bullet\sqrt{1-\rho_\bullet^2}, \mu_\bullet - \rho_\bullet\sigma_\bullet, 0),$$

$$\phi^*(u,t_0;\sigma_\bullet,\mu_\bullet,\rho_\bullet) = \phi^*(ue^{\int_t^T \sigma_s\mu_s\rho_s\,ds}, t_0; \sigma_\bullet\sqrt{1-\rho_\bullet^2}, \mu_\bullet - \rho_\bullet\sigma_\bullet, 0) + \sigma_{t_0}\rho_{t_0}.$$

Proof. Take $n=1$ and $\gamma_t = (\rho_t)$, $\Gamma_t = (\sqrt{1-\rho_t^2})$ in Theorem 3.1. □

We note that ρ_t near 1 corresponds to transformed claim volatility small, i.e. transformed claim nearly constant, which corresponds intuitively to a nearly complete market.

3.2 Expected Shortfall Hedging

In this section we consider the problem of minimising the expected shortfall in a hedge, $\mathbb{E}[(C_T - X_T)^+]$. We shall show that in our context this reduces to the problem of maximising the probability of a perfect hedge.

A general survey of risk measures including both shortfall-based and quantile-based measures is given in Artzner et al [2]. The suitability of shortfall-based risk measures is discussed in Acerbi and Tasche [1]. Robustness to the choice of loss function in shortfall-based risk measures is considered in Favero and Vargiolu [14].

Suppose that as before we have our contingent claim and our traded-asset price process revealed as geometric Brownian motions,

$$\frac{dC_t}{C_t} = \sigma_t dW_t + \kappa_t dt, \quad C_0 = c_0 \tag{3.8}$$

$$\frac{dP_t}{P_t} = \rho_t dW_t + \sqrt{1-\rho_t^2}\,dB_t + \mu_t dt, \tag{3.9}$$

with a strategy ϕ_t chosen by the agent determining how the wealth evolves,

$$\frac{dX_t^\phi}{X_t^\phi} = \phi_t\frac{dP_t}{P_t}, \quad X_0 = x_0. \tag{3.10}$$

We shall denote the minimal expected shortfall by $V_E(x,c,t) = V_E(x,c,t;\sigma_\bullet,\mu_\bullet,\rho_\bullet)$.

Theorem 3.2 *For the model dynamics (3.8)-(3.9) we have*

$$V_E(x,c,t_0) = c\left(1 - V\left(\frac{x}{c}e^{\int_{t_0}^T (\sigma_s^2)\,ds}, t_0; \sigma_\bullet, \mu_\bullet + \rho_\bullet\sigma_\bullet, \rho_\bullet\right)\right) \tag{3.11}$$

$$= c\left(1 - V\left(\frac{x}{c}e^{\int_{t_0}^T (\sigma_t\rho_t\mu_t - (1-\rho_t^2)\sigma_t^2)\,dt}, t_0; \sigma_\bullet\sqrt{1-\rho_\bullet^2}, \mu_\bullet, 0\right)\right). \tag{3.12}$$

Proof. If we take $d\mathbb{Q}/d\mathbb{P} = C_T/\mathbb{E}C_T$ then under $\mathbb{Q}$ we have the following dynamics

$$\frac{dC_t}{C_t} = \sigma_t d\beta_t + (\sigma_t^2 + \kappa_t)dt \tag{3.13}$$

$$\frac{dP_t}{P_t} = \rho_t d\beta_t + \sqrt{1-\rho_t^2}\,dB_t + (\mu_t + \sigma_t^2)dt, \tag{3.14}$$

where $d\beta_t = dW_t - \sigma dt$ gives a standard Brownian motion under $\mathbb{Q}$. Now the value function for the expected-shortfall problem has

$$V_E(x,c,t) = \inf_\phi \mathbb{E}\left[C_T \left(1 - \frac{X_T^\phi}{C_T}\right)^+ \middle| X_t = x, C_t = c\right]$$

$$= c \inf_\phi \mathbb{E}_\mathbb{Q}\left[(1-U_T^\phi)^+ \middle| X_t = x, C_t = c\right],$$

evaluating $\mathbb{E}[C_T]$.

Hence, since $(1-\xi)^+ = 1 - (\xi \wedge 1)$,

$$\frac{V_E(x,c,t)}{c} = 1 - \sup_\phi \mathbb{E}_\mathbb{Q}\left[U_T^\phi \wedge 1 \middle| X_t = x, C_t = c\right]$$

$$= 1 - V\left(\frac{x}{c}e^{\int_t^T \sigma_s^2\,ds}, t; \sigma_\bullet, \mu_\bullet + \rho_\bullet\sigma_\bullet, \rho_\bullet\right),$$

using a simple transformation to adjust for the drift in C under $\mathbb{Q}$. Finally using Corollary 3.1 to remove the correlation we obtain (3.12). □

We see that the results for quantile hedging have direct analogues for expected shortfall hedging. Although we have worked with one tradeable asset here, it is clear that Theorem 3.2 would extend directly to the case of multiple assets.

In the constant claim case we have that

$$V_E(x,c,t;0,\mu,0) = c\left(1 - V\left(\frac{x}{c}, t; 0, \mu, 0\right)\right)$$

$$= c\left(1 - \Phi\left(\Phi^{-1}\left(\frac{x}{c}\right) + |\mu|\sqrt{T-t}\right)\right).$$

Further, in the zero-drift case we have

$$
\begin{aligned}
V_E(x,c,t;\sigma,0,\rho) &= c\left(1 - V\left(\frac{x}{c}e^{\sigma\sqrt{(1-\rho^2)(T-t)}}, t; \sigma\sqrt{1-\rho^2}, 0, 0\right)\right) \\
&= c\left(1 - \Phi\left(\Phi^{-1}\left(\frac{x}{c}e^{\sigma\sqrt{(1-\rho^2)(T-t)}}\right) + \sigma\sqrt{(1-\rho^2)(T-t)}\right)\right). \quad (3.15)
\end{aligned}
$$

A plot of (3.15), for $c = 2$ and $\sigma = 1$, is given in Figure 3-1. We see that this tends to zero for very large wealths, x, as we would expect. We also note that, for $x < c$, there is a sharp increase in the expected shortfall when there is very little time-to-go. This is plausible as even a little bit of trading time can be very useful.

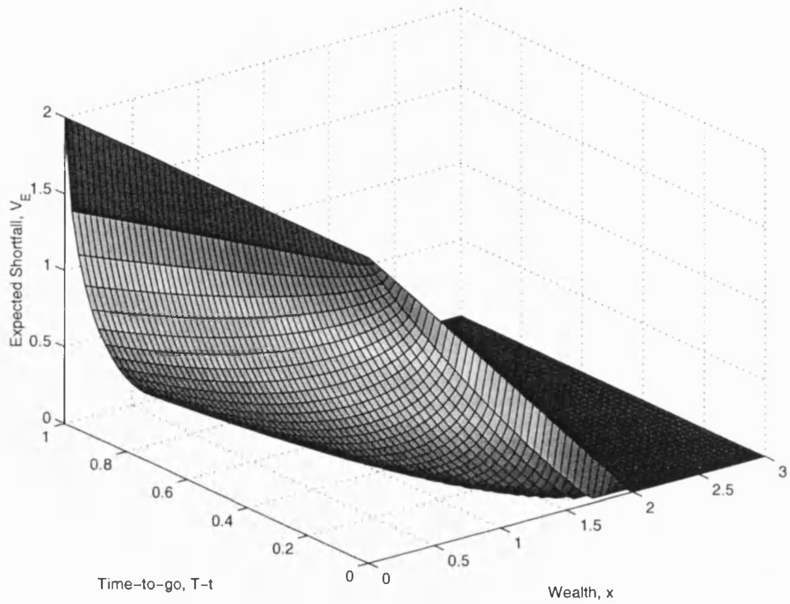

Figure 3-1: The Minimal Expected Shortfall when $c = 2$, $\mu = 0$ and $\sigma = 1$

Chapter 4

An Independent Claim: The Numerical Approach

We begin by briefly recapping on what we have done so far and considering why we will need to turn to numerical techniques in this chapter.

4.1 Consolidating Remarks

We recall that we are concerned with an agent who is obliged to meet a contingent claim at a fixed time in the future. The agent wishes to maximise the chance of being able to meet the claim, that is the quantile hedging problem.

We obtained, in Chapter 2, a full solution to the problem where the market is complete, but the agent has insufficient funds to exactly replicate the claim.

In Section 3.1 we showed that our most general problem, where the contingent claim is partially correlated with n tradeable assets, can be reduced to the problem where the claim is independent of a single tradeable asset. Furthermore in Section 3.2 we saw that, in our context, the problem of minimising the expected shortfall can be recast as a quantile hedging problem with modified parameters. Consequently in this chapter we focus attention on the case of quantile heding with an independent claim.

In the case of an independent claim and an asset with zero drift we obtained explicit formulae for the probability of a perfect hedge by verifying the optimality of putative optimal policies. It is not clear what one would conjecture in the case of an independent claim and tradeable asset with non-zero drift so in this case we turn to numerical techniques to solve the HJB equation we have for the value function.

4.2 An Asset with Zero Drift Revisited

To confirm the reliability of our numerical techniques we first apply them to the case where the tradeable asset has zero drift, for which we already have explicit formulae. Our HJB equation in this case is

$$V_u u + \frac{1}{2} V_{uu} u^2 - \frac{1}{\sigma^2} \dot{V} = 0.$$

In order to remove the dependence on the wealth ratio, u, in the coefficients of this equation we make the transformation $Y(w,t) = V(e^w, t)$ which gives

$$Y_w + Y_{ww} - \frac{2}{\sigma^2} \dot{Y} = 0. \tag{4.1}$$

The boundary conditions become $Y(w,0) = e^w \wedge 1$, $Y(w,t) \to 1$ as $w \to \infty$ and $Y(w,t) \to 0$ as $w \to -\infty$.

We solve this problem numerically using a finite difference scheme. We take a finite region of the domain we wish to solve the problem over and divide it up into a grid of N_t by N_w points. At each point on this grid we consider the relationship between the value function at that point and at surrounding points implied by discretising the derivatives in the partial differential equation. This gives a system of simultaneous equations. We approximate our time derivatives with backward differences, that is using the point under consideration and the point one time step back in our grid. By doing this we can solve the system of equations one row at a time. We shall take our space (i.e. wealth ratio) derivatives to be an average from the current row and the preceding row. This is called the Crank-Nicholson scheme and is more stable than the alternatives of using just the current row for space derivatives (the explicit scheme) or just the preceding row (the implicit scheme).

Take $1 \le n_t \le N_t$, $1 \le n_w \le N_w$, $h_w = (w_{\max} - w_{\min})/N_w$ and $h_t = T/N_t$. We discretise (4.1) by replacing Y_w by

$$\frac{1}{2}\left(\frac{Y_{n_t,n_w+1} - Y_{n_t,n_w-1}}{2h_w}\right) + \frac{1}{2}\left(\frac{Y_{n_t-1,n_w+1} - Y_{n_t-1,n_w-1}}{2h_w}\right),$$

Y_{ww} by

$$\frac{1}{2}\left(\frac{Y_{n_t,n_w+1} - 2Y_{n_t,n_w} + Y_{n_t,n_w-1}}{h_w^2}\right) + \frac{1}{2}\left(\frac{Y_{n_t-1,n_w+1} - 2Y_{n_t-1,n_w} + Y_{n_t-1,n_w-1}}{h_w^2}\right),$$

and $\dot{Y}$ by

$$\frac{Y_{n_t,n_w} - Y_{n_t-1,n_w}}{h_t}$$

This gives us the following system of linear equation for the n_tth row of our grid

$$A\begin{pmatrix} Y_{n_t,1} \\ \\ \vdots \\ \\ Y_{n_t,N_w} \end{pmatrix} = \begin{pmatrix} Y_{n_t,1} \\ Y_{n_t-1,2}\left(-\frac{2}{\sigma^2 h_t}\right) - \frac{1}{2}M_2 \\ Y_{n_t-1,3}\left(-\frac{2}{\sigma^2 h_t}\right) - \frac{1}{2}M_3 \\ \vdots \\ Y_{n_t,N_w} \end{pmatrix},$$

where A is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\left(-\frac{1}{2h_w}+\frac{1}{h_w^2}\right) & \left(-\frac{1}{h_w^2}-\frac{2}{\sigma^2 h_t}\right) & \frac{1}{2}\left(\frac{1}{2h_w}+\frac{1}{h_w^2}\right) & 0 & 0 \\ 0 & \frac{1}{2}\left(-\frac{1}{2h_w}+\frac{1}{h_w^2}\right) & \left(-\frac{1}{h_w^2}-\frac{2}{\sigma^2 h_t}\right) & \frac{1}{2}\left(\frac{1}{2h_w}+\frac{1}{h_w^2}\right) & 0 \\ & & & & \ddots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$M_{n_w} = \frac{Y_{n_t-1,n_w+1} - Y_{n_t-1,n_w-1}}{2h_w} + \frac{Y_{n_t-1,n_w+1} - 2Y_{n_t-1,n_w} + Y_{n_t-1,n_w-1}}{h_w^2}.$$

Our transformed problem covers a region stretching to infinity in both spatial (i.e. wealth ratio) directions. When we restrict this to a finite region to apply the finite difference scheme we no longer have expressions for the boundary conditions. On the small-wealth-ratio boundary we find that $Y(w_{\min}, t) = 0$ is a satisfactory approximation. However, we find we need to take more care with the boundary condition on the large-wealth-ratio boundary.

The strategy of doing nothing, even at horizon time, has success probability

$$\mathbb{P}(C_T \leq x_t | C_t) = \Phi\left(\frac{\log u + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).$$

We shall use this lower bound as our value function along the large-wealth-ratio boundary

$$Y(w_{\max}, t) = \Phi\left(\frac{w + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right).$$

The error in the numerical calculation is shown in Figure 4-1 for $\sigma = 1$. We note that this error is very small. The largest error occurs along the line $u = 1$. This is propagated back from the non-smooth point on the time-horizon boundary condition.

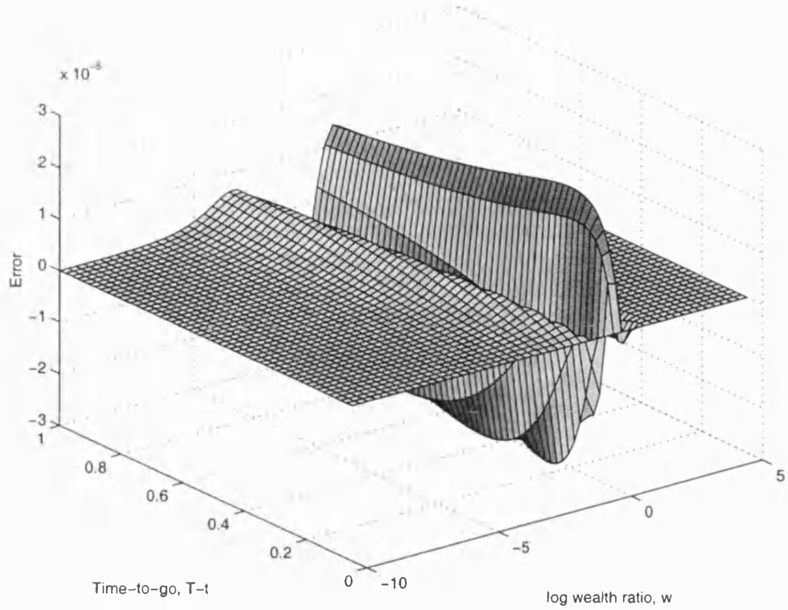

Figure 4-1: The Error in Calculating the Value Function using the Crank-Nicholson scheme when $\sigma = 0$, $\mu = 1$

4.3 An Asset with Non-zero Drift: Methods

In this case the HJB equation for the value function is

$$V_u u + \frac{1}{2} V_{uu} u^2 - \frac{\mu^2 V_u^2}{2\sigma^2 V_{uu}} - \frac{1}{\sigma^2}\dot{V} = 0. \tag{4.2}$$

Again we can remove the dependence on u in the coefficients by making the transformation $Y(w,t) = V(e^w, t)$ which gives

$$Y_w + Y_{ww} - \frac{\mu^2}{\sigma^2}\frac{Y_w^2}{(Y_{ww} - Y_w)} - \frac{2}{\sigma^2}\dot{Y} = 0.$$

However, it is clear that discretising this gives a non-linear system of simultaneous equations.

Recall though that if the optimal policy is ϕ^* then the transformed value function satisfies

$$\frac{1}{2} Y_{ww}\left((\phi^*)^2 + \sigma^2\right) + Y_w\left(\mu\phi^* + \frac{1}{2}\sigma^2 - \frac{1}{2}(\phi^*)^2\right) - \dot{Y} = 0. \tag{4.3}$$

(Substituting

$$\phi^* = \frac{Y_w \mu}{Y_w - Y_{ww}} \tag{4.4}$$

recovers (4.2)).

Now (4.3) is linear and so, using the techniques of Section 4.2, given any policy ϕ we can obtain the success probability corresponding to using that policy. This allows us to use a policy improvement scheme. Given an initial policy, ϕ^0, we can obtain its success probability function, Y^0, then using a discretised from of (4.4) we can use this to obtain an improved policy, ϕ^1 and so on. A reasonable choice for initial policy is the optimal policy in the $\mu = 0$ case, that is $\phi^0 = 0$.

Recall that the Crank-Nicholson scheme that we are using to solve our partial differential equations is stable because it uses an averaged difference to approximate space (i.e. wealth-ratio) derivatives. We also do this in our approximation to the derivatives in (4.4). Figure 4-2 shows the policy resulting from 20 steps of this scheme, with $\sigma = 1$ and $\mu = 0.1$. We find the difference between the 19th and 20th policy is generally around 1×10^{-4}. In comparison to the sizes of the policies themselves this is very small so we conclude that the scheme has converged.

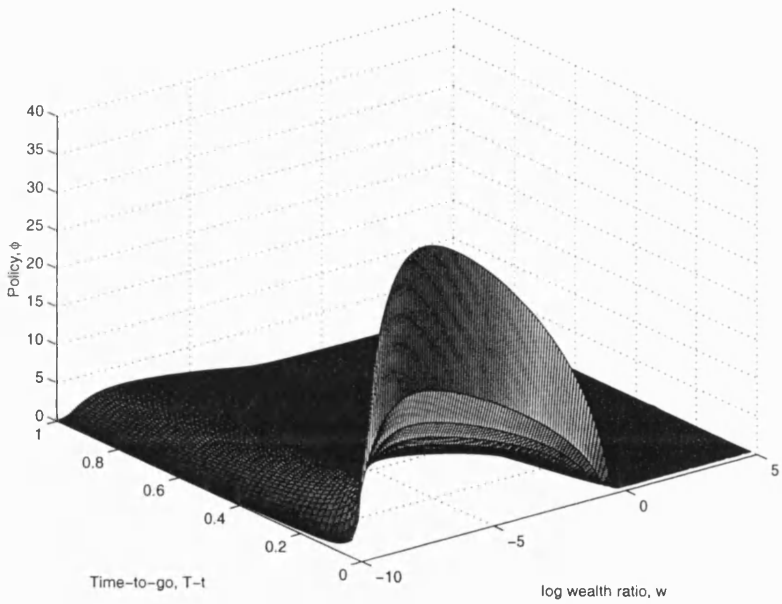

Figure 4-2: The Policy Obtained from 20 steps of Policy Improvement when $\sigma = 1$, $\mu = 0.1$

We saw in, Section 2.5, that with the policy $\phi^0 = 0$ we obtain a success probability

$V^0(u,t)$ given by (2.16). As before we have, $Y^0(w,t) = V^0(e^w,t)$. Now

$$\begin{aligned}\phi_t^1 &= \frac{-\mu Y_w^0}{Y_{ww}^0 - Y_w^0} \\ &= \mu\sigma\sqrt{t}\left(1 - \Phi\left(\frac{w + \frac{3}{2}\sigma^2 t}{\sigma\sqrt{t}}\right)\right)\sqrt{2\pi}\exp\left\{\frac{1}{2}\left(\frac{w + \frac{3}{2}\sigma^2 t}{\sigma\sqrt{t}}\right)^2\right\},\end{aligned}$$

which is not finite as $w \to -\infty$. This is intuitively plausible: as our capital diminishes to nothing we take increasingly large positions in the hope of meeting the claim. However, we note that we do not see this in Figure 4-2. This suggests that we need to use a better boundary condition for our calculation of the value functions along the small-wealth-ratio boundary.

In Section 2.6 we obtained a policy $\hat{\phi}$, the optimal fixed policy, which although not optimal is a sensible guess at an effective policy. We can obtain the value function from using this policy and see what improvement it offers over the policy of doing nothing until immediately before the time horizon, which was optimal in the case of a zero-drift asset. This value improvement, $\hat{Y} - Y^0$, is shown in Figure 4-3.

We note that this improvement is quite small. It tends to zero for both large and small wealth ratios as we would expect. As the time to go decreases so does the value improvement, with less time-to-go there is less opportunity to benefit from the presence of drift. However the improvement also decreases for large time-to-go. We are using a policy which was chosen as if no future changes in policy were allowed which will become more sub-optimal as there is more time-to-go.

We shall use this value function for the small-wealth-ratio boundary condition.

4.4 An Asset with Non-zero Drift: Results

In Section 4.3 we explained the methods we use to obtain the optimal value improvement, $Y - Y^0$, in the case of an independent claim and an asset with non-zero drift, that is the difference between the success probability following the strategy of doing nothing until horizon time and the success probability following the optimal strategy. Figure 4-4 gives a plot of this value improvement in the case where $\sigma = 1$ and $\mu = 0.1$.

As with the value improvement from using the optimal fixed policy the improvement is small, it decreases to zero for both large and small wealth ratio and it decreases when there is little time to go. However, as we would expect we do not find the value improvement decreasing for large time-to-go as we did with the policy that was only optimal amongst fixed policies.

The optimal policy, ϕ^* is shown in Figure 4-5. As with the optimal fixed policy, $\hat{\phi}$,

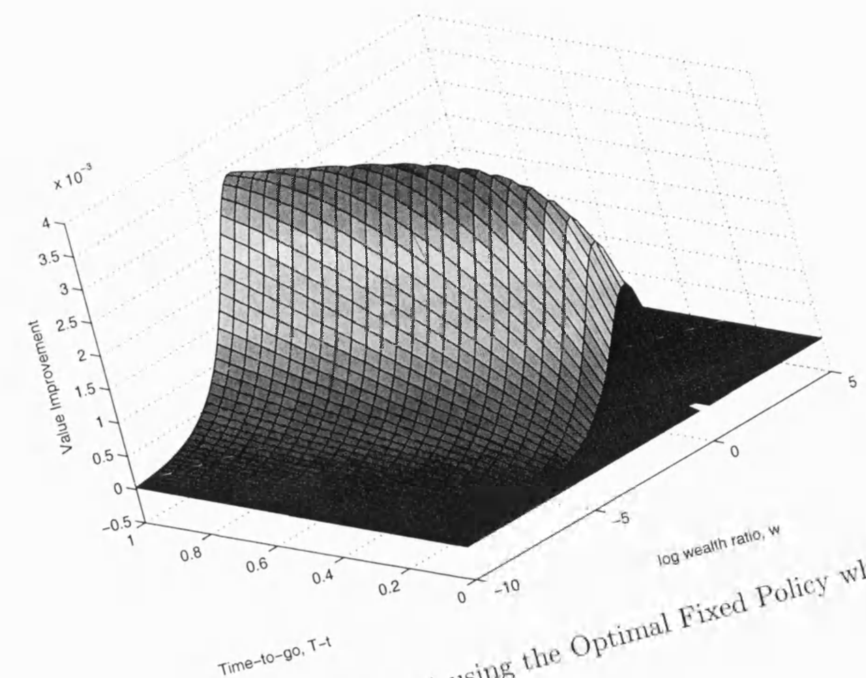

Figure 4-3: The Value Improvement using the Optimal Fixed Policy when $\sigma = 1$ and $\mu = 0.1$

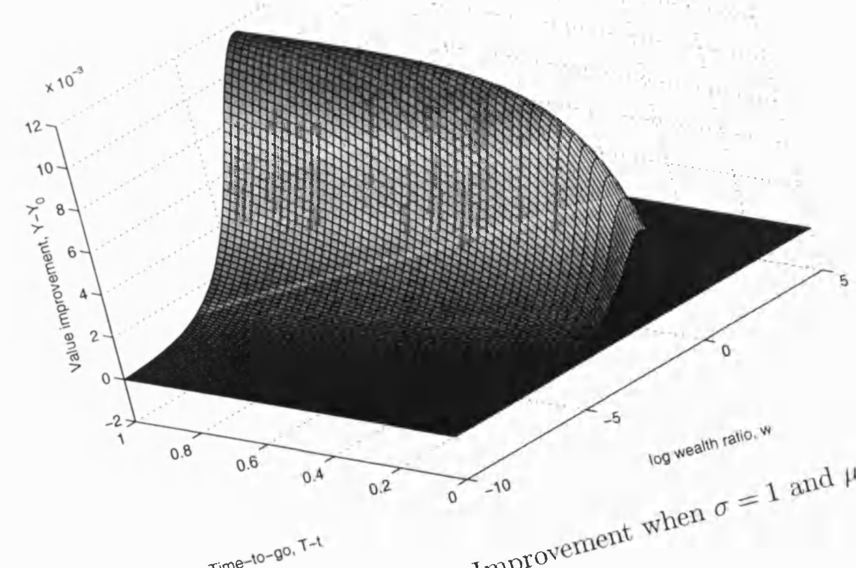

Figure 4-4: The Optimal Value Improvement when $\sigma = 1$ and $\mu = 0.1$

there is trading throughout time though, as we would expect, more trading near the horizon time. We note again that with much time-to-go there is a much less marked increase in trading for decreasing wealth ratio.

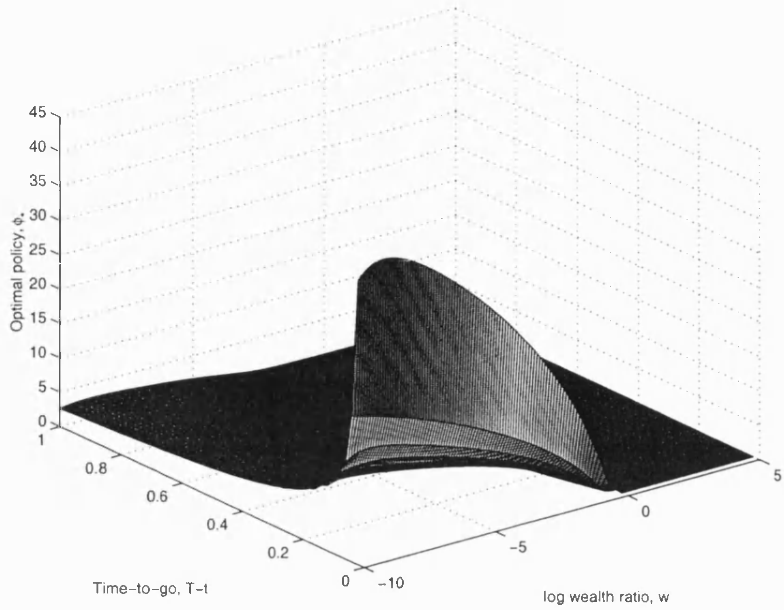

Figure 4-5: The Optimal Policy when $\sigma = 1$ and $\mu = 0.1$

We have already noted that the value improvements we have seen have been small but we also note that we have been considering quite small values of μ. Consider now how the value improvement varies with μ. Figure 4-6 shows a plot of log maximal value improvement (i.e. $\log \max_{(w,t)}(Y(w,t) - Y^0(w,t))$) against log asset drift (i.e. $\log \mu$). This indicates that, over the values of μ considered, the value improvement varies essentially like μ^2.

Consider formally writing Y as a power series in μ

$$Y = Y^0 + \frac{\partial Y}{\partial \mu}(0)\mu + \frac{\partial^2 Y}{\partial \mu^2}(0)\mu^2 + O(\mu^3).$$

However, our problem is invariant under $\mu \mapsto -\mu$ so, assuming they exist,

$$\frac{\partial Y}{\partial \mu}(0) = 0,$$

and

$$\frac{\partial^3 Y}{\partial \mu^3}(0) = 0.$$

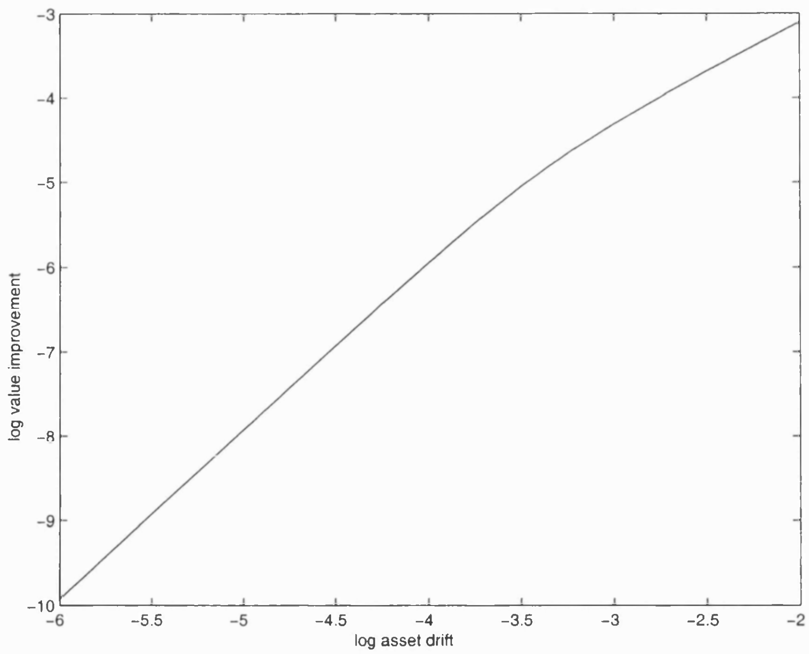

Figure 4-6: Regression of log Maximal Value Improvement against log Asset Drift

Hence, formally,

$$Y(\mu) = Y^0 + \check{Y}\mu^2 + O(\mu^4),$$

for some $\check{Y}$. So we can write our optimal policy thus

$$\begin{aligned}\phi^* &= \frac{\mu Y_w}{Y_w - Y_{ww}} \\ &= \frac{\mu Y_w^0}{Y_w^0 - Y_{ww}^0} + O(\mu^3).\end{aligned}$$

Substituting for the optimal policy ϕ^* in our HJB equation (4.3) for Y gives

$$\frac{1}{2}\sigma^2(\check{Y}_w + \check{Y}_{ww}) - \dot{\check{Y}} = \frac{-(Y_w^0)^2}{2(Y_w^0 - Y_{ww}^0)} + O(\mu^2),$$

after simplifying using the HJB equation for Y^0. Using this we can write $\check{Y}$ as an expectation, and then apply Fubini's Theorem to give

$$\check{Y}(w,t) = \int_0^t \int_{-\infty}^{\infty} \frac{\gamma(y, t-s)}{\sqrt{2\pi}\sigma\sqrt{s}} \exp\left\{ -\frac{y - w - \frac{1}{2}\sigma^2 s}{2\sigma\sqrt{s}} \right\} dy\, ds,$$

where

$$\gamma(w,t) = \frac{(Y_w^0)^2}{2(Y_w^0 - Y_{ww}^0)}$$

We see that this is not finite since as $y \to -\infty$ the exponential term grows faster than $\gamma(y, t-s)$ decays. However, the form of our formal power series does give a partial explanation for the μ^2 behaviour of $\check{Y}$.

Figure 4-7 shows the proportion (in the case $\sigma = 1$, $\mu = 0.1$) of the optimal value improvement that is achieved through using the optimal fixed policy. (This plot is restricted to the region where the value improvements are sufficiently bounded away from zero for this calculation to be meaningful). We see that it achieves a fairly consistent proportion between 0.5 and 0.9 of the value improvement. We do notice though, that for wealth ratio near 1 and small time-to-go the fixed policy does relatively badly. We expect this as it is not able to respond to last minute variations which can push the wealth ratio into or out of the success region.

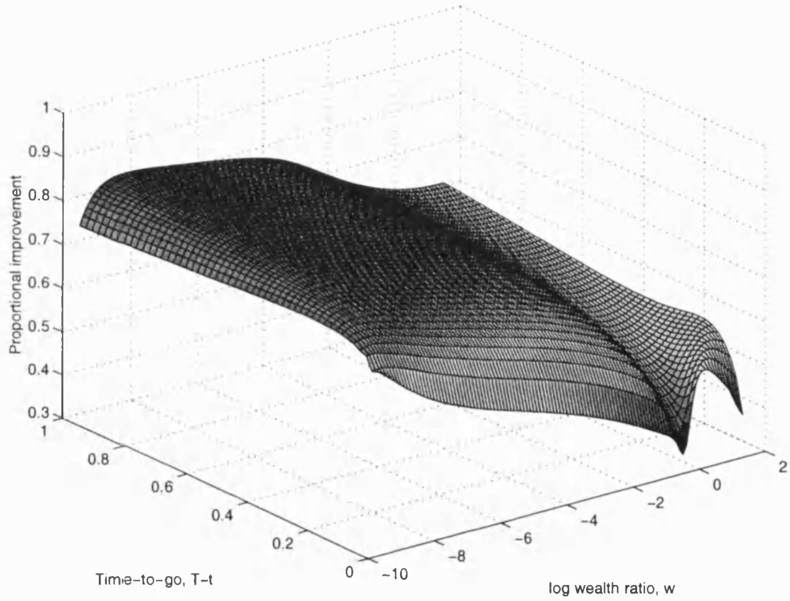

Figure 4-7: The Proportion of the Maximal Improvement Achieved by the Optimal Fixed Policy when $\sigma = 1$ and $\mu = 0.1$

4.5 A Short-Sales Constraint

We consider briefly what happens if we impose a constraint on the short sale of stock, that is if we require that the policy (ϕ_t) is always positive. The motivation behind this is that in actual financial markets short selling often entails logistical and regulatory difficulties.

Recalling our expression (2.6) for the optimal policy,

$$\phi_t^* = \frac{V_u(\mu - \sigma\rho)}{V_{uu}u} - \sigma\rho,$$

we see that when there is no correlation, $\rho = 0$, a short-sales constraint either has no effect, if $\mu > 0$, or reduces the policy to the trivial strategy of holding no stock, if $\mu < 0$. So we are only concerned with the case with non-zero correlation. However, Corollary 3.1 reduces the problem with non-zero correlation to one with zero correlation but modified parameters and a modified policy,

$$\phi_t \mapsto \phi_t - \sigma\rho.$$

So it is enough to consider the case with zero correlation and the policy constrained to be no lower than some constant value,

$$\phi_t \geq K.$$

This constraint is very easily applied to the numerical scheme described in this chapter. Figure 4-8 shows the difference in success probabilities between the unconstrained problem, $K = 0$, and a constrained problem with $K = 0.5$. We see that the constraint is most disadvantageous when our wealth is roughly equal to the value of the contingent claim. It has little effect when we have very small or very large wealth ratio as then we are either very likely to lose or very likely to win even if we cannot use exactly the policy we wanted. We also notice that the difference in success probability decreases as the time-to-go decreases. As there is less time remaining it is less of a problem that our choice of policy is being constrained as we want to take a large policy then anyway.

Figure 4-9 shows the difference in policy in the same two cases. On the right we see a flat region where we have been forced to have the minimal stock holding of $K = 0.5$ when we wanted to hold no stock, as our wealth already significantly exceeded the claim. On the left we see that our policy is unaffected by the constraint when we have little wealth. In the middle we see that when our wealth is roughly equal to the value of the claim we have a slightly smaller stock holding. This compensates for the excess stock holding for large wealth.

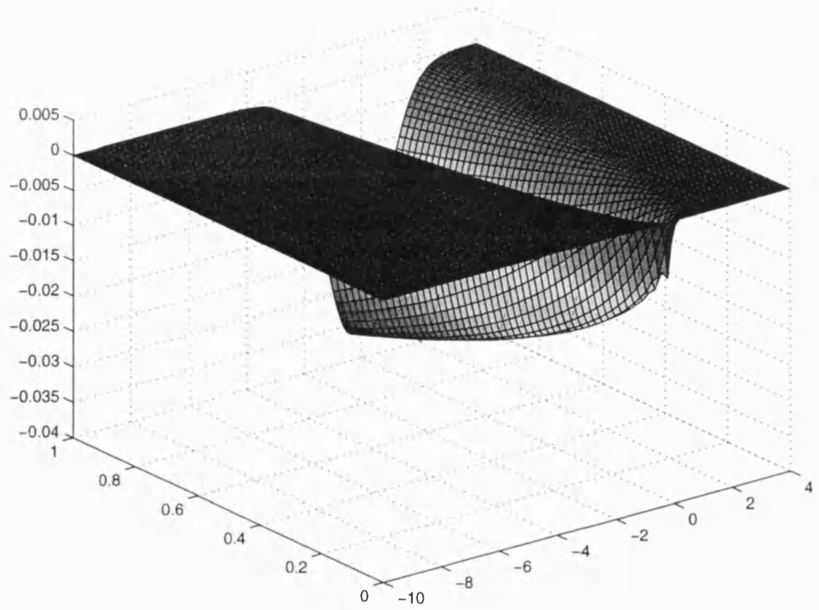

Figure 4-8: The Difference in Success Probability between the Unconstrained Problem and a Constrained Problem with $K = 0.5$ when $\sigma = 1$ and $\mu = 0.1$

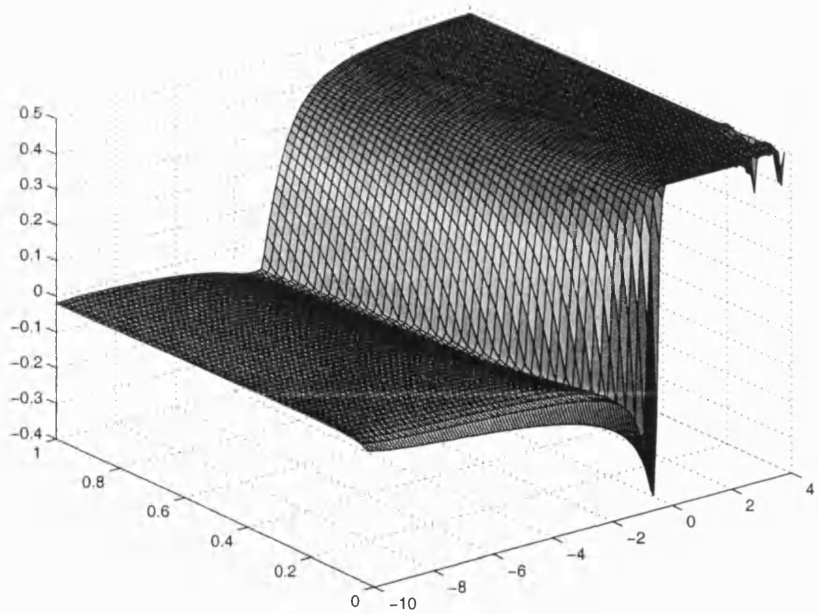

Figure 4-9: The Difference in Policy between the Unconstrained Problem and a Constrained Problem with $K = 0.5$ when $\sigma = 1$ and $\mu = 0.1$

Chapter 5

An American Problem

This chapter is concerned with the problem of maximising the chance of reaching an upper boundary before a lower boundary and before a time horizon. It is American in the sense that once the upper boundary is reached the claim is called-in.

If at any time we have wealth equal to or greater than the value of the claim we can call in the claim and pay it off. In such circumstances our optimal success probability is

$$V_A(u,t) = \sup_{\phi_s : t \leq s \leq T} \mathbb{P}\left(X_s^{\phi} \geq C_s \text{ some } s \in [t,T] \right),$$

recalling that u is the current wealth ratio X_t/C_t.

This problem was posed by Karatzas [33] in the case of a constant claim. We should note though that he imposed a no-short-sales constraint, $\phi \geq 0$, on the portfolio choices, of the sort we considered in Section 4.5. We do not impose such a constraint here.

As before we have geometric Brownian price process and claim dynamics,

$$\frac{dC_t}{C_t} = \sigma dW_t, \quad C_0 = c_0 \tag{5.1}$$

$$\frac{dP_t}{P_t} = dB_t + \mu dt. \tag{5.2}$$

The price process is independent of the claim. The strategy ϕ_t is chosen by the agent,

$$\frac{dX_t^{\phi}}{X_t^{\phi}} = \phi_t \frac{dP_t}{P_t}, \quad X_0 = x_0. \tag{5.3}$$

5.1 An Asset with Zero Drift

When the asset has zero drift we can obtain the value function analytically. We essentially turn an American option problem into a barrier option problem.

Proposition 5.1 *Given the wealth and claim dynamics (5.1) - (5.3), with $\mu = 0$, we have success probability in the American case given by*

$$V_A(u,t) = 1 - f_C\left(1, u, -2\sigma^2, T-t, \sigma\right) + \frac{1}{u} f_P\left(1, \frac{1}{u}, \sigma^2, T-t, \sigma\right),$$

where we denote by $f_P(s,k,r,T,\sigma)$ and $f_C(s,k,r,T,\sigma)$ the Black-Scholes prices of a European put and respectively European call with strike k and maturity T on a stock with price s and volatility σ under interest rate r.

Proof. We follow a similar approach to that in Proposition 2.5. We hypothesise that the optimal strategy is to do nothing until just before T, then if we have not met the claim at any stage so far we trade so as to have wealth C_T with probability x_0/C_T and have zero wealth otherwise.

We expect this will be optimal since there is no asset drift which we can take advantage of and so it is best to wait until we have as much information about the claim as possible. Furthermore, by doing nothing until the horizon we increase the probability that the claim will have drifted below our initial wealth level and if this happens we can call in the claim.

This strategy, call it (ψ_t), gives a success probability of

$$\mathbb{P}\left(X_s^{\psi} \geq C_s \text{ some } s \in [0,T]\right) = \mathbb{E}\left[\mathbb{1}_{\{\underline{C}_T \leq x_0\}} + \frac{x_0}{C_T}\mathbb{1}_{\{\underline{C}_T > x_0\}}\right],$$

where

$$\underline{C}_t = \inf_{0 \leq s \leq t} C_s.$$

That is the success probability is equal to the value of a barrier option with payoff

$$1 - \left(1 - \frac{x_0}{C_T}\right)\mathbb{1}_{\{\underline{C}_T > x_0\}},$$

Now

$$C_T = c_0 e^{\sigma W_T - \frac{1}{2}\sigma^2 T} = c_0 e^{\sigma W_T^*},$$

where $W_t^* = W_t - \frac{1}{2}\sigma t$ and $\mathbb{P}^*$ is such that W_t^* is a standard Brownian motion. Hence

$$\begin{aligned}
\mathbb{P}\left(X_s^\psi \geq C_s \text{ some } s \in [t,T]\right) &= 1 - \mathbb{E}^*\left[\left(1 - \frac{x_0}{c_0}e^{-\sigma W_T^*}\right)\mathbb{1}_{\left\{c_0 e^{\sigma \underline{W}_T^*} \geq x_0\right\}}\frac{d\mathbb{P}}{d\mathbb{P}^*}\right] \\
&= 1 - \int_\alpha^\infty \left(1 - \frac{x_0}{c_0}e^{-\sigma y}\right)\mathbb{P}^*\left(W_T^* \in dy,\, \underline{W}_T^* > \alpha\right)\frac{d\mathbb{P}}{d\mathbb{P}^*}\, dy,
\end{aligned}$$

where $\alpha = \frac{1}{\sigma}\log\frac{x_0}{c_0}$. Since

$$\mathbb{P}^*\left(W_T^* \geq y,\, W_T^* > \alpha\right) = \mathbb{P}^*\left(W_T^* \geq y\right) - \mathbb{P}^*\left(W_T^* \geq y,\, \underline{W}_T^* < \alpha\right),$$

we have that

$$\mathbb{P}^*\left(W_T^* \in dy,\, \underline{W}_T^* > \alpha\right) = \frac{1}{\sqrt{2\pi T}}\left(\exp\left\{-\frac{y^2}{2T}\right\} - \exp\left\{-\frac{(y-2\alpha)^2}{2T}\right\}\right).$$

Further,

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{-\left(-\frac{1}{2}\sigma\right)W_T - \frac{1}{2}\left(-\frac{1}{2}\sigma\right)^2 T},$$

and so

$$\frac{d\mathbb{P}}{d\mathbb{P}^*} = e^{-\frac{1}{2}\sigma W_T + \frac{1}{8}\sigma^2 T} = e^{-\frac{1}{2}\sigma W_T^* - \frac{1}{8}\sigma^2 T}.$$

Hence

$$\begin{aligned}
&\mathbb{P}\left(X_s^\psi \geq C_s \text{ some } s \in [t,T]\right) \\
&\quad = 1 - \int_\alpha^\infty \frac{1}{\sqrt{2\pi T}}\left(1 - \frac{x_0}{c_0}e^{-\sigma y}\right)\left(\exp\left\{-\frac{y^2}{2T}\right\} - \exp\left\{-\frac{(y-2\alpha)^2}{2T}\right\}\right) \\
&\qquad \times \exp\left\{-\frac{1}{2}\sigma y - \frac{1}{8}\sigma^2 T\right\}\, dy.
\end{aligned}$$

Gathering together terms we have

$$\begin{aligned}
&\mathbb{P}\left(X_s^\psi \geq C_s \text{ some } s \in [t,T]\right) \\
&\quad = 1 - \int_\alpha^\infty \frac{1}{\sqrt{2\pi T}}\left(\exp\left\{-\frac{\left(y + \frac{1}{2}\sigma T\right)^2}{2T}\right\} - \exp\left\{-\frac{\left(y - 2\alpha + \frac{1}{2}\sigma T\right)^2}{2T} - \alpha\sigma\right\}\right) dy \\
&\qquad + \frac{x_0}{c_0}\int_\alpha^\infty \frac{1}{\sqrt{2\pi T}}\left(\exp\left\{-\frac{\left(y + \frac{3}{2}\sigma T\right)^2}{2T} + \sigma^2 T\right\}\right. \\
&\qquad\qquad \left. - \exp\left\{-\frac{\left(y - 2\alpha + \frac{3}{2}\sigma T\right)^2}{2T} - 3\alpha\sigma + \sigma^2 T\right\}\right) dy,
\end{aligned}$$

which simplifies to

$$\begin{aligned}
&\mathbb{P}\left(X_s^{\psi} \geq C_s \text{ some } s \in [t,T]\right) \\
&\quad = 1 - \Phi\left(\frac{-\log\frac{x_0}{c_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + \frac{c_0}{x_0}\Phi\left(\frac{\log\frac{x_0}{c_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\
&\qquad + \frac{x_0}{c_0}e^{\sigma^2 T}\Phi\left(\frac{-\log\frac{x_0}{c_0} - \frac{3}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) - \left(\frac{c_0}{x_0}\right)^2 e^{\sigma^2 T}\Phi\left(\frac{\log\frac{x_0}{c_0} - \frac{3}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).
\end{aligned}$$

Differentiation and substitution confirms that this satisfies the HJB equation (2.7). Substituting $t = T$ and $u = 1$ give the required boundary conditions of $u \wedge 1$ and 1 respectively. So this is indeed the value function. Rearranging gives the required result. □

5.2 An Asset with Non-zero Drift

We can obtain the value function in the case with non-zero drift by exactly the method we used in Chapter 4. The time-horizon and small-wealth boundary conditions,

$$V_A(0,t) = 0, \quad V_A(u,T) = u \wedge 1,$$

are essentially the same as in the quantile hedging problem. The large-wealth boundary condition is slightly more straightforward than in the European problem,

$$V_A(1,t) = 1.$$

Figure 5-1 shows the difference between the success probabilities in the American and European cases when $\sigma = 1$ and $\mu = 0.1$. As we would expect this is very small except near the large wealth boundary. Unless our wealth is of the same order of magnitude as the claim we are unlikely to succeed by the claim value falling below our wealth and our calling the claim in. Near the boundary the difference decreases from about 3×10^{-2} down to zero as time-to-go decreases. With less time-to-go there is less opportunity for the claim to fall below our current wealth and so less opportunity for us to succeed by using the American feature of the claim.

Figure 5-2 shows the value improvement in the American case with $\sigma = 1$ and $\mu = 0.1$. We notice that this is qualitatively very similar to the European case. It is also of a similar magnitude, having a maximum of about 1×10^{-2}. It is somewhat surprising that the value improvements are qualitatively very similar given that they

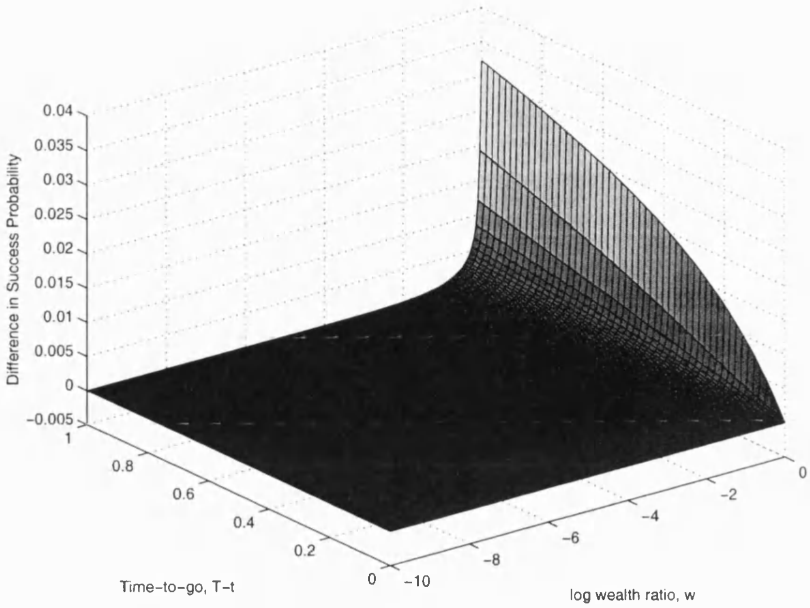

Figure 5-1: The Difference in Success Probability between the American and European Cases when $\sigma = 1$ and $\mu = 0.1$

are noticeably smaller in magnitude than the difference between probabilities in the European case and in the American case.

Figure 5-3 shows that the difference in policy between the American and European cases is very small especially compared with the magnitude of the policies. Along the large wealth boundary we have a slightly larger stock holding in the American case. It is worth taking a small risk to increase our wealth even when maturity of the claim is some time off since we can call the claim in at any point. We also have a slightly larger stock holding just before the maturity of the claim, though this is proportionally much less significant. As we can call the claim in at any time we increase our risk exposure slightly faster.

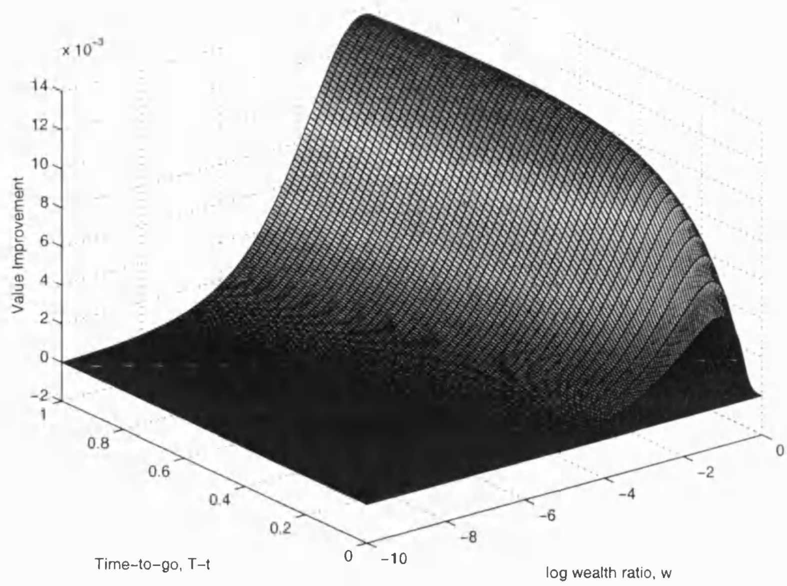

Figure 5-2: The Value Improvement in the American case when $\sigma = 1$ and $\mu = 0.1$

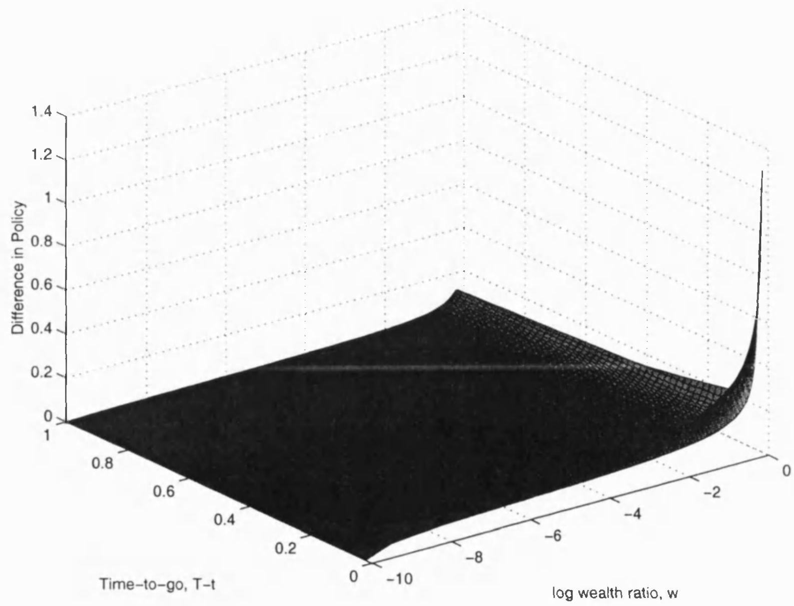

Figure 5-3: The Difference in Policy between the American and European cases when $\sigma = 1$ and $\mu = 0.1$

Chapter 6

The Value of Timely Information

In Chapter 1 we considered the quantile hedging problem in the case where the contingent claim was not revealed until the time horizon, T. In this chapter we apply such ideas to the problem where the claim has a log-normal distribution but is revealed instantaneously at some time γ. We are interested in how the success probability depends on γ.

The results of Chapter 1 apply directly to the case where the contingent claim is revealed at time T, that is at the maturity of the claim. Similar ideas apply to the case where the claim is revealed just before maturity, the case where the claim is revealed just after time 0, the beginning of trading, and the case where the claim is revealed at some point in between. For the sake of comparison we will always take the claim to have the same distribution.

In the case of constant relative risk aversion there is a much higher level of tractability. In addition to results on the timing of instantaneously revealed claims, we are able to obtain results on the continuously-revealed case given different timings of claim volatility.

6.1 Quantile Hedging

In Chapter 2 we had a claim revealed continuously with dynamics,

$$\frac{dC_t}{C_t} = \sigma dW_t, \quad C_0 = c_0.$$

Suppose now that we have a claim revealed at maturity but given by

$$C = c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T},$$

where G is a $N(0,1)$ random variable. This corresponds to the distribution of the process in Chapter 2 given the information available at time 0. The distribution function is

$$F_C(y) = \Phi\left(\frac{\log \frac{y}{c_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right).$$

We shall write

$$k(\xi, T) = \sqrt{2\sigma^2 T - 2\log\left(\xi c_0 \sigma \sqrt{2\pi T}\right)}.$$

Obtaining the concave relaxation of F_C and inverting gives, after some simplification,

$$I(\xi) = \mathbb{1}_{\{\xi<\tilde{\xi}\}} c_0 e^{-\frac{3}{2}\sigma^2 T} \exp\left\{\sigma\sqrt{T}\, k(\xi, T)\right\},$$

where $\tilde{\xi}$ is given by

$$\Phi\left(k(\tilde{\xi}, T) - \sigma\sqrt{T}\right) = \frac{1}{\sigma\sqrt{T}}\Phi'\left(k(\tilde{\xi}, T) - \sigma\sqrt{T}\right).$$

We saw, in Chapter 1, that it is optimal to take our terminal wealth to be

$$X_T^* = I(\lambda Z_T).$$

where $Z_T = \left(d\tilde{\mathbb{P}}/d\mathbb{P}\right)\Big|_{\mathcal{F}_T}$ and $\tilde{\mathbb{P}}$ is the equivalent measure under which the price process is a martingale.

Our tradeable asset is independent of the claim and has the geometric-Brownian dynamics we considered in earlier chapters,

$$\frac{dP_t}{P_t} = dB_t + \mu dt,$$

where B is a Brownian motion independent of C, so

$$Z_T = e^{-\mu B_T - \frac{1}{2}\mu^2 T}.$$

The Lagrange multiplier λ satisfies

$$\mathbb{E}[e^{-\mu B_T - \frac{1}{2}\mu^2 T} I(\lambda e^{-\mu B_T - \frac{1}{2}\mu^2 T})] = x_0.$$

We can evaluate such expectations by numerical quadrature then solve for λ by binary search.

We shall denote by $V^\gamma(x_0/c_0)$ the success probability, given initial wealth ratio

x_0/c_0, with a claim revealed at time γ. So we have

$$
\begin{aligned}
V^T\left(\frac{x_0}{c_0}\right) &= \mathbb{P}\left(X_T^* \geq C_T\right) \\
&= \mathbb{P}\left(I\left(\lambda e^{-\mu B_T - \frac{1}{2}\mu^2 T}\right) \geq c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T}\right),
\end{aligned}
$$

(recalling that the function I depends on x_0, c_0 and T) which again can be evaluated by quadrature.

In Figure 6-1 we plot, in the case $\sigma = 1$ and $\mu = 0.1$, the optimal success probability when the claim is revealed instantaneously at maturity, minus the optimal success probability when the claim is revealed continuously. We see that for moderate wealth the continuous revelation of information gives a substantial improvement. Close inspection shows that as the time-to-go increases the improvement increases and then decreases again. This is intuitively plausible as the extra time means that more information is being made available but then with even more time this is less relevant as the claim could drift far from its current position.

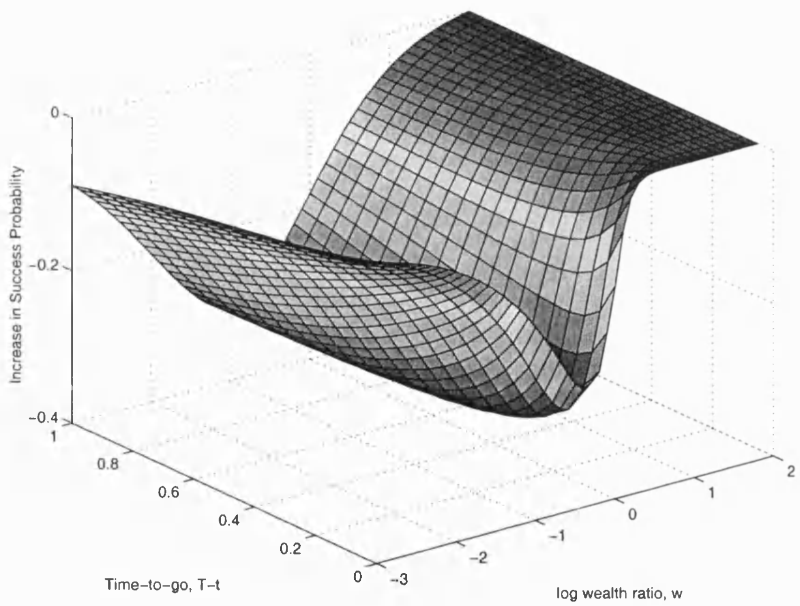

Figure 6-1: The Increase in Success Probability when the Contingent Claim is Instantaneously Revealed after the Horizon rather than Continuously Revealed

Now consider the case where the contingent claim is still random,

$$
C = c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T},
$$

but is revealed just after the outset, at time 0^+. Proposition 2.2 tells us that given a claim value of $C = c$, the optimal success probability is

$$\sup_{\phi_t:0\le t\le T} \mathbb{P}\left(X_T^\phi \ge C_T \middle| X_0 = x_0,\, C = c\right) = \Phi\left(\Phi^{-1}\left(\frac{x_0}{c}\right) + |\mu|\sqrt{T}\right).$$

So before the outset the success probability is

$$\begin{aligned} V^{0^+}\left(\frac{x_0}{c_0}\right) &= \sup_{\phi_t:0\le t\le T} \mathbb{P}\left(X_T^\phi \ge C_T \middle| X_0 = x_0\right) \\ &= \mathbb{E}\left[\Phi\left(\Phi^{-1}\left(\frac{x_0}{c_0}e^{-\sigma\sqrt{T}G+\frac{1}{2}\sigma^2 T}\right) + |\mu|\sqrt{T}\right)\right], \end{aligned}$$

We have that

$$\begin{aligned} &\mathbb{P}\left(\Phi\left(\Phi^{-1}\left(\frac{x_0}{c_0}e^{-\sigma\sqrt{T}G+\frac{1}{2}\sigma^2 T}\right) + |\mu|\sqrt{T}\right) \le y\right) \\ &\quad = \mathbb{P}\left(e^{-\sigma\sqrt{T}G+\frac{1}{2}\sigma^2 T} \le \frac{c_0}{x_0}\Phi\left(\Phi^{-1}(y) - |\mu|\sqrt{T}\right)\right) \end{aligned}$$

so

$$V^{0^+}\left(\frac{x_0}{c_0}\right) = \int_0^1 \left(1 - \Phi\left(\frac{\log\left(\Phi\left(\Phi^{-1}(y) - |\mu|T\right)\right) - \log\frac{x_0}{c_0} - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right) dy,$$

which we can evaluate by numerical quadrature.

Next suppose that the contingent claim is revealed just before maturity, that is at time T^-. Unlike the case where the claim is revealed at maturity, time T, we are able to trade in response to the claim, if only for an instant. Consequently the result of Theorem 2.1 applies. So we wish to maximise

$$\sup_{X_T^\phi} \mathbb{E}\left[\frac{X_T^\phi}{C_T} \wedge 1\right],$$

subject to the budget constraint

$$\tilde{\mathbb{E}}\left[X_T^\phi\right] = x_0.$$

Now we can take a similar approach to that in Chapter 1,

$$\sup_{X_T^\phi} \mathbb{E}\left[\frac{X_T^\phi}{C_T} \wedge 1\right] = \sup_{X_T^\phi} \mathbb{E}\left[f_C\left(X_T^\phi\right)\right],$$

where

$$f_C(x_0) = \mathbb{E}\left[\frac{x_0}{C_T} \wedge 1\right].$$

So the Lagrangian for the problem is

$$L(X,\lambda) = \mathbb{E}[f_C(X) - \lambda(ZX - x_0)] = \int_\Omega \left(f_C\left(X(\omega)\right) - \lambda\left(Z(\omega)X(\omega) - x_0\right)\right) \mathbb{P}(d\omega).$$

Suppose f_C is concave, so we can take $I = (f'_C)^{-1}$ and then we have that the optimal terminal wealth is

$$X^*_T = I(\lambda Z).$$

In the case we are considering here we have

$$\begin{aligned} f_C(x_0) &= \mathbb{E}\left[\frac{x_0}{c_0} e^{-\sigma W_T + \frac{1}{2}\sigma^2 T} \wedge 1\right] \\ &= \mathbb{E}\left[\mathbb{1}_{\left\{\frac{x_0}{c_0} e^{-\sigma W_T + \frac{1}{2}\sigma^2 T} \geq 1\right\}} + \frac{x_0}{c_0} e^{-\sigma W_T + \frac{1}{2}\sigma^2 T} \mathbb{1}_{\left\{\frac{x_0}{c_0} e^{-\sigma W_T + \frac{1}{2}\sigma^2 T} \leq 1\right\}}\right] \end{aligned}$$

which becomes

$$f_C(x_0) = \mathbb{P}\left(\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T \leq \log\frac{x_0}{c_0}\right) + \frac{x_0}{c_0}\mathbb{E}\left[\mathbb{1}_{\left\{\frac{x_0}{c_0} e^{-\sigma\sqrt{T}G + \frac{3}{2}\sigma^2 T} \leq 1\right\}}\right],$$

where the second term comes from a change of measure of the sort used in the probabilistic derivation of the Black-Scholes formula. So

$$f_C(x_0) = \Phi\left(\frac{\log\frac{x_0}{c_0} + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) + \frac{x_0}{c_0}\left(1 - \Phi\left(\frac{\log\frac{x_0}{c_0} + \frac{3}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right).$$

Differentiating and simplifying gives

$$f'_C(x_0) = \frac{1}{c_0}\left(1 - \Phi\left(\frac{\log\frac{x_0}{c_0} + \frac{3}{2}\sigma^2 T}{\sigma\sqrt{T}}\right)\right).$$

Differentiating again shows us that f_C is concave. Inverting the expression for f_C we obtain

$$I(\xi) = c_0 \exp\left\{\sigma\sqrt{T}\Phi^{-1}\left(1 - c_0\xi\right) - \frac{3}{2}\sigma^2 T\right\} \mathbb{1}_{\left\{\xi \leq \frac{1}{c_0}\right\}}.$$

We can now evaluate

$$V^{T^-}\left(\frac{x_0}{c_0}\right) = \mathbb{E}\left[I\left(\lambda e^{-\mu B_T - \frac{1}{2}\mu^2 T}\right) e^{-\sigma W_T + \frac{1}{2}\sigma^2 T} \wedge 1\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{y_1} \frac{e^{-\frac{1}{2}(y_1^2+y_2^2)}}{2\pi} \left(I\left(\lambda e^{-\mu\sqrt{T}y_1 - \frac{1}{2}\mu^2 T}\right) e^{-\sigma\sqrt{T}y_2 + \frac{1}{2}\sigma^2 T} \wedge 1\right) dy_2 dy_1, \tag{6.1}$$

by numerical quadrature after having obtained λ by binary search as before.

Figures 6-2, 6-3 and 6-4 show comparisons of the success probability for the claim revealed: at maturity T, just before maturity T^-, and just after the outset 0^+, in the case with $\sigma = 1$, $\mu = 0.1$. For comparison we also show the success probability when the claim is continuously revealed, in the case $\sigma = 1$, $\mu = 0.1$ and in the case $\sigma = 1$, $\mu = 0$.

The other parameter values are initial wealth $x_0 = e^{-0.1}$ and mean claim value $c_0 = 1$. The choice of initial wealth was such that the probability of success through the strategy of doing nothing at all is neither close to zero nor close to one. Notice that as we vary T we vary the the distribution of C since

$$C = c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T}.$$

We note that the success probabilities all lie in the order of increasing information availability and all converge to the initial wealth ratio x_0/c_0. In the continuously-revealed case, the success probability is lower when the drift is zero than when it is non-zero. In the case with zero drift the optimal strategy is to do nothing until maturity. Although there is just as much information as for non-zero drift, one cannot make full use of it.

The plots of success probability are all initially decreasing in maturity, as C becomes more variable, but then increasing as $C \downarrow 0$ for large maturity.

A striking feature of Figure 6-2 is that the bottom line is far below any of the others. It is an order of magnitude more useful to be able to trade, if only for an instant, in response to the value of the claim than it is to observe the claim through time.

Figure 6-3 is a close-up of Figure 6-2 where the line corresponding to the claim being revealed after maturity has been omitted. Starting from the bottom, we see that the bottom two lines are essentially exactly the same. So the success probability is the same, up to slight numerical inaccuracies, for the problem with claim revealed just before maturity and for the problem where the claim is revealed continuously but the tradeable asset has zero drift. In the latter case the optimal strategy is to hold no stock until just before the horizon time. The lack of drift in the tradeable asset means that

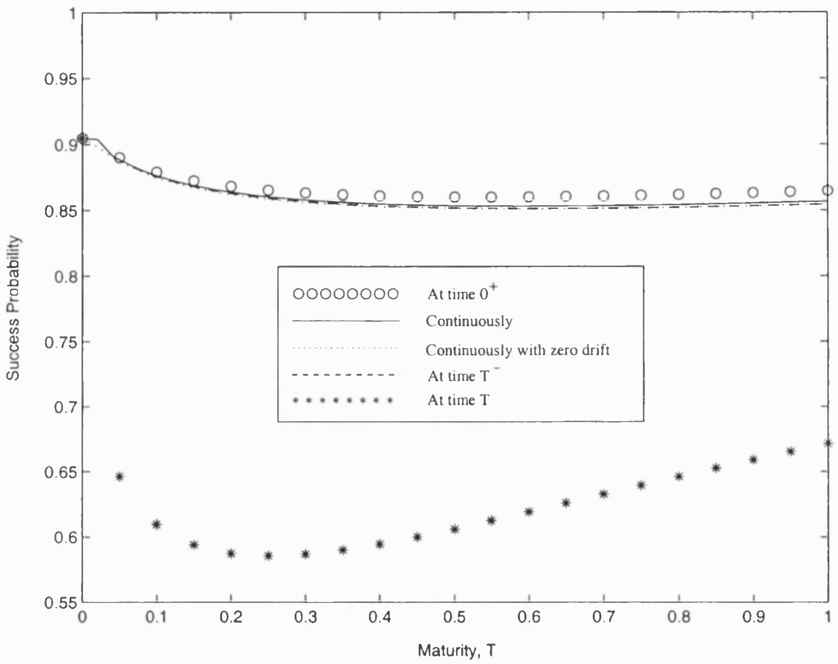

Figure 6-2: Comparison of Optimal Success Probabilities for varying Maturity when $x_0 = e^{-0.1}$, $c_0 = 1$

we are unable to take advantage of the information about the claim value that is being revealed through time.

The gap between these lines and the third one up corresponds to the increase in success probability resulting from receiving information about the evolution of the claim value through time and being able to use this information. The gap between the third line up and the top line corresponds to the increase in success probability resulting from knowing the exact claim value from the outset.

We see that the second gap is a bit more than twice the size of the first. This is not surprising, we would expect it be more useful to know the claim value from the outset than to be able to respond to it changing through time.

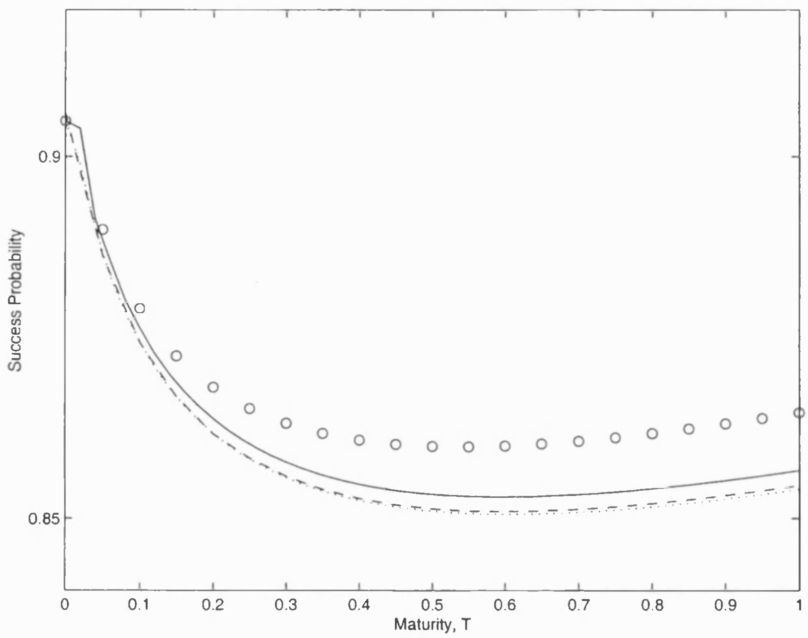

Figure 6-3: Close-Up of Comparison of Optimal Success Probabilities for varying Maturity when $x_0 = e^{-0.1}$, $c_0 = 1$

Figure 6-4 has $T = 1$ and $c_0 = 1$, with initial wealth, x_0, varying. We notice that apart from for very large and very small values of wealth the bottom line is separated from the other lines by a substantial gap. The benefit from being able to trade in response to the claim value is substantial. All of the lines converge to 0 for very small wealth and 1 for very large wealth. (This would have been even more apparent if the log wealth axis had been taken to be wider, but then it would have been difficult to distinguish which line was which.) This feature of the graph corresponds to near certain failure for very small wealth and near certain success for wealth an order of magnitude or more larger than the expected claim value, $c_0 = 1$.

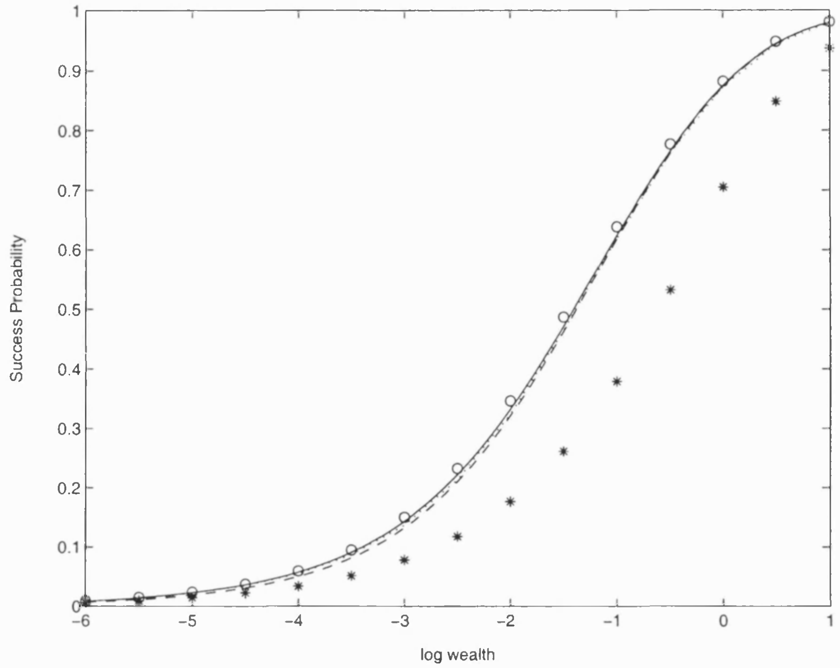

Figure 6-4: Comparison of Optimal Success Probabilities for varying Initial Wealth when $T = 1$, $c_0 = 1$

Finally we turn to the situation where the claim is revealed at an intermediate time strictly between the outset and the horizon, that is time γ with $0 < \gamma < T$. As before the claim has distribution

$$C = c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T}.$$

We can obtain the HJB equation for the success probability before the claim is revealed, i.e. up to time γ, in the same manner as we did in Chapter 2

$$0 = \sup_{\phi} \left\{ \frac{1}{2} V_{xx} x^2 \phi^2 + V_x x \mu \phi \right\} + \dot{V}.$$

Indeed this is the same HJB equation (2.5) but with $\sigma^2 = 1$, and $c_0 = 1$. The boundary condition for the time the claim is revealed, that is time γ, follows in exactly the same manner as for the success probability (6.1) in the case where the contingent claim is revealed just before the time horizon. So if we have wealth ratio x/c_0 at time γ our success probability is

$$\mathbb{E}\left[\Phi\left(\Phi^{-1}\left(\frac{x}{c_0}e^{-\sigma\sqrt{T}G+\frac{1}{2}\sigma^2 T}\right)+|\mu|\sqrt{T-\gamma}\right)\right].$$

This can again be evaluated by numerical quadrature. Having done this we can apply the Crank-Nicholson scheme in the manner we did in Chapter 4.

Figure 6-5 shows the success probability benefit, when $T = 1$ with $x_0 = e^{-0.1}$, $c_0 = 1$, from having the claim revealed at an intermediate time γ, as a proportion of the difference in success probability for the cases with claim revealed just after the outset, time 0^+, and that with claim revealed just before the time horizon, time T^-. That is it shows

$$\frac{V^{\gamma}(e^{-0.1}) - V^{T^-}(e^{-0.1})}{V^{0^+}(e^{-0.1}) - V^{T^-}(e^{-0.1})}.$$

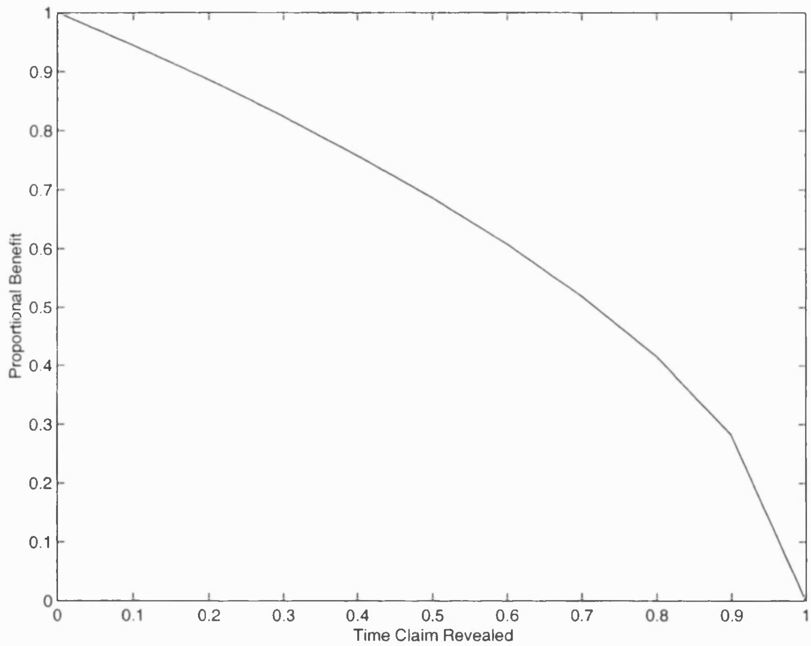

Figure 6-5: The Proportional Benefit in Success Probability for a Claim Revealed at Intermediate Time when $T = 1$, $x_0 = e^{-0.1}$, $c_0 = 1$

We see that this is concave and indeed is quite sharply decreasing with decreasing time-to-go near the time horizon. That is the marginal benefit of knowing the value of the contingent claim a little earlier is greatest just before the time horizon.

We briefly consider how the value function in the continuously-revealed case depends on the timing of volatility. In Section 2.4 we considered the complete market case, that is the situation when the contingent claim and the tradeable asset are perfectly correlated. If we have time-dependent volatility (σ_t) and constant asset drift μ the success probability is

$$V(u,0) = \Phi\left(\Phi^{-1}\left(ue^{\mu\int_0^T \sigma_s\,ds}\right) + \sqrt{\int_0^T (\sigma_s - \mu)^2\,ds}\right).$$

Fixing $\int_0^T \sigma_s^2\,ds = (\sigma^*)^2 T$ and writing the success probability as a function of $\Sigma = \int_0^T \sigma_s\,ds$, we wish to maximise

$$\tilde{V}(\Sigma) = \Phi^{-1}\left(ue^{\mu\Sigma}\right) + \sqrt{(\sigma^*)^2 T + \mu^2 - 2\mu\Sigma}.$$

That we can write the success probability as a function of Σ shows that we are not concerned about the timing of volatility only about the total volatility.

For given asset drift μ, we can find the optimal distribution of volatility Σ^* as a function of the wealth ratio u. This is shown in Figures 6-6, 6-7 and 6-8 for the case $\sigma^* = 1$, $T = 1$ with asset drifts of $\mu = 0.5$, $\mu = 1$ and $\mu = 1.5$ respectively.

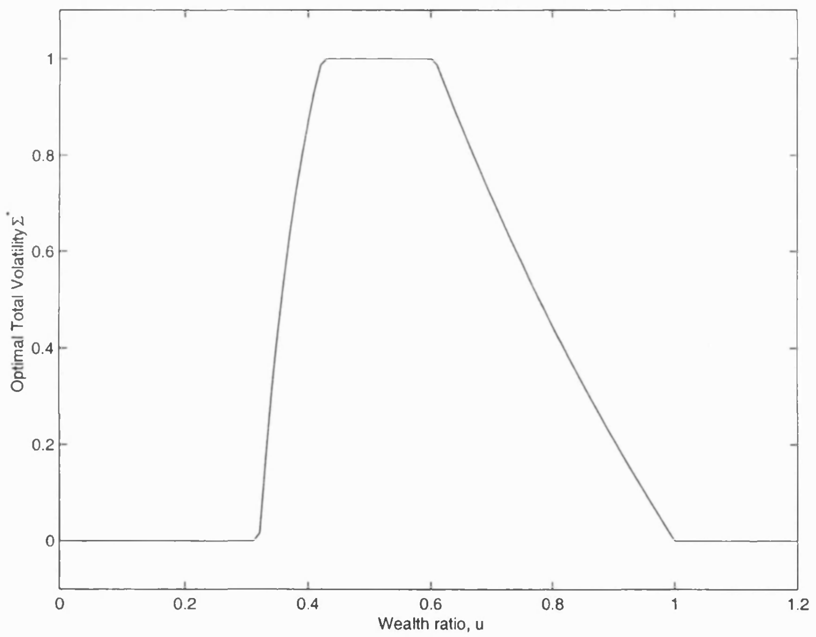

Figure 6-6: The Optimal Distribution of Volatility as a Function of Wealth Ratio u when $(\sigma^*)^2 T = 1$ and $\mu = 0.5$

We see that, as u increases, we initially have a flat region with $\Sigma^* = 0$ then it increases up to 1, though the increase is less sudden for μ near 1. After this Σ^* decays away from 1, reaching 0 at $u = 1$. If our wealth ratio is very small or very large we

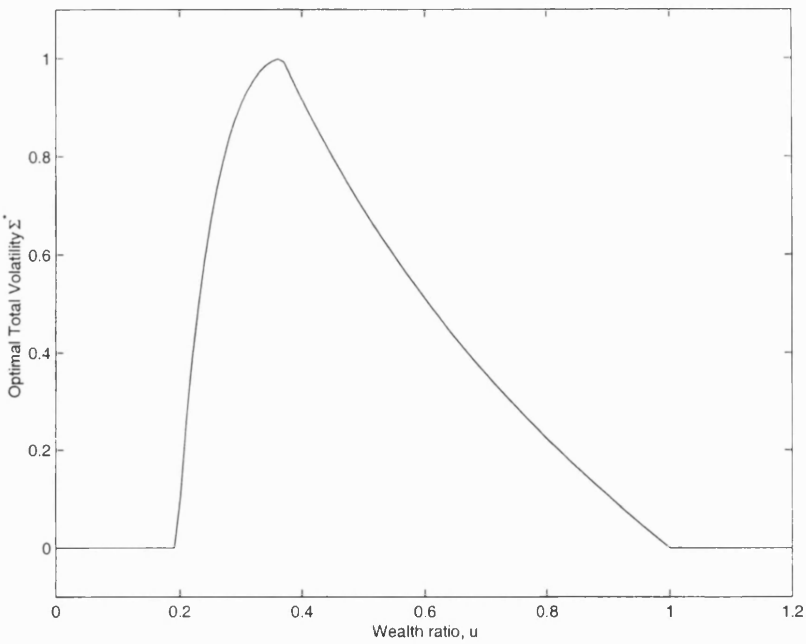

Figure 6-7: The Optimal Distribution of Volatility as a Function of Wealth Ratio u when $(\sigma^*)^2T = 1$ and $\mu = 1$

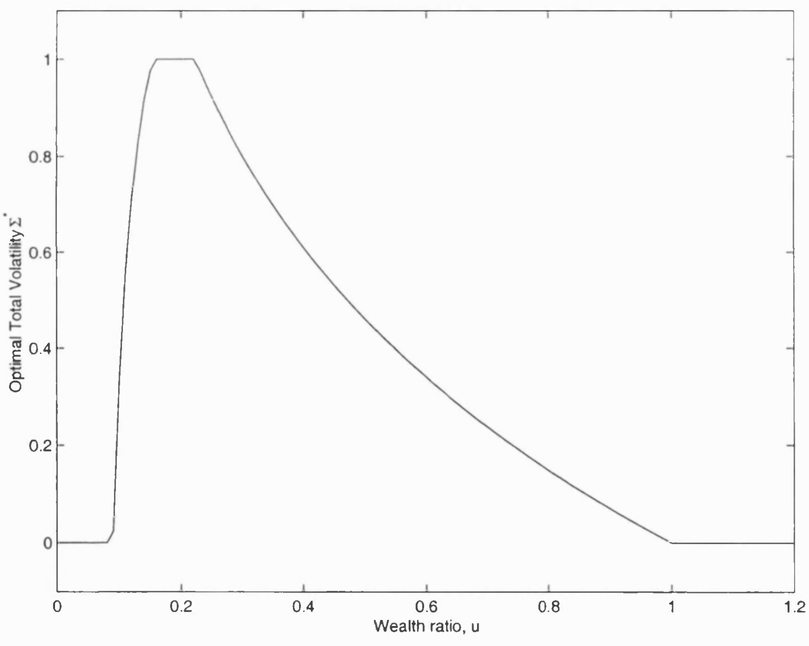

Figure 6-8: The Optimal Distribution of Volatility as a Function of Wealth Ratio u when $(\sigma^*)^2T = 1$ and $\mu = 1.5$

would prefer to have the volatility of the claim occur as a spike, but for intermediate values of the wealth ratio we prefer constant volatility. This intermediate flat region is large for μ far from 1 but for $\mu = 1$ it is a single point.

We can explain why Σ^* decreases away from 1 for larger μ as follows. $\tilde{V}$ contains the term $\Phi^{-1}\left(ue^{\mu\Sigma}\right)$. Since $\Phi^{-1}(x) \uparrow \infty$ as $x \uparrow 1$, taking Σ such that $e^{\mu\Sigma}$ is greater than $1/u$ will not increase $\tilde{V}$. Consequently, we have that as u increases beyond $e^{-\mu}$ the largest value we would want to take for Σ^* decreases from 1. We notice that in all three figures the decline in Σ^* on the right hand side of the graph begins at $e^{-\mu}$ (this being at around 0.607, 0.368 and 0.223 respectively).

Now consider the region where $u < e^{-\mu}$. Differentiating $\tilde{V}$ we have

$$\tilde{V}'(\Sigma) = \frac{\mu u e^{\mu\Sigma}}{\Phi'\left(\Phi^{-1}\left(ue^{\mu\Sigma}\right)\right)} - \frac{\mu}{\sqrt{1+\mu^2-2\mu\Sigma}}.$$

Suppose that $\tilde{V}$ has at most one turning point in $[0,1]$. Then it is optimal to take $\Sigma^* = 0$ when u is less than the value that gives $\tilde{V}'(0) = 0$. Similarly it is optimal to take $\Sigma^* = 1$ when u is greater than the value that gives $\tilde{V}'(1) = 0$. So we anticipate that curves demarcating regions with $\Sigma^* = 0$, $0 < \Sigma^* < 1$ and $\Sigma^* = 1$ will be given by

$$\frac{u}{\Phi'\left(\Phi^{-1}(u)\right)} = \frac{1}{\sqrt{1+\mu^2}},$$

and

$$\frac{u}{\Phi'\left(\Phi^{-1}\left(ue^{\mu}\right)\right)} = \frac{e^{-\mu}}{|1-\mu|},$$

respectively. We notice the first curve is symmetric about $\mu = 0$ and the second curve touches the first at $\mu = 0$. We can obtain the optimal total volatility, Σ^*, numerically. When we do this we find that the regions where $\Sigma^* = 0$, $0 < \Sigma^* < 1$ and $\Sigma^* = 1$ do indeed correspond to the regions we had anticipated.

Figure 6-9 shows Σ^* for varying u and μ. The white regions correspond to where $\Sigma^* = 0$, the black regions corresponds to $\Sigma^* = 1$. The grey regions correspond to intermediate values, $0 < \Sigma^* < 1$.

Let us consider the economic interpretation of this. When $\mu = 1$ the contingent claim is a martingale under the pricing measure and so the expression for the success probability simplifies greatly. This is why we find that we have distinct behaviour for $\mu = 1$.

We notice that crossing the line $\mu = 0$ black regions change to white and vice versa. This makes sense as we change from having a long position in the stock to a short position or vice versa. Consequently we might expect our preference for constant or

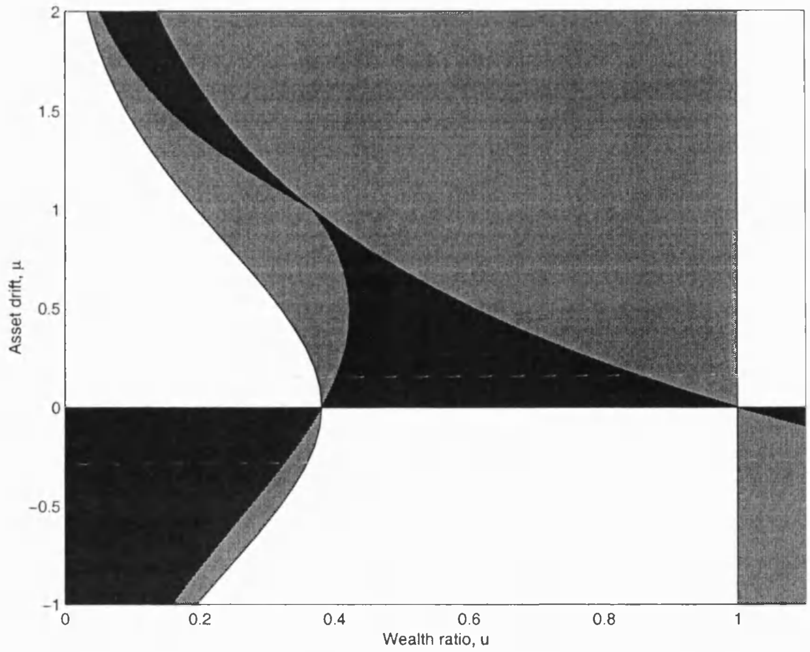

Figure 6-9: The Optimal Distribution of Volatility as a Function of Wealth Ratio and Asset Drift μ

spikey volatility, $\Sigma^* = 0$ or $\Sigma^* = 1$, to change. On the other hand, as u varies the behaviour is smooth in that white changes to black via grey and vice versa.

As we cross the line $u = 1$, that is as we change from having less money than we need to meet the claim to having more, the behaviour changes. For positive μ, we change from intermediate distribution of volatility to taking volatility spikey. If we have more money than we need we want sudden jumps in the claim value that we can then respond to rather than constant change in the claim value. For negative μ the opposite is true.

We also see a change in behaviour as we cross the line $u = e^{-\mu}$. We saw in Proposition 2.4 that the factor $e^{\mu \int_t^T \sigma_s \, ds}$ reduces the general complete market case to the constant claim case so we would expect on crossing this line to change from taking our claim volatility constant to taking a more varied distribution of volatility.

So it remains to consider the region where $\mu > 0$ and $u < e^{-\mu}$. For very small wealth we find that, as was the case when we had very large wealth, we prefer to have sudden jumps in the claim value which we respond to over time rather than having constant change. As u increases this effect diminishes, with the desired Σ^* changing smoothly to 1.

6.2 Constant Relative Risk Aversion

Consider the more tractable problem where the value function is that corresponding to constant relative risk aversion. That is at maturity, for wealth ratio u we receive utility

$$\frac{u^{1-R}}{1-R}.$$

In this more tractable setting we are able to consider different schedules of claim volatility for a continuously revealed claim and relate this to different times at which a claim might be instantaneously revealed. We shall see that, surprisingly, the success probability does not depend on the timing of volatility.

First consider the case where the claim is revealed at maturity, that is at time T, after all trading has ended. At time T^{-} our expected utility for given wealth x is

$$\begin{aligned} f_C(x) &= \frac{1}{1-R}\mathbb{E}\left[\left(\frac{x}{C_T}\right)^{1-R}\right] \\ &= \frac{x^{1-R}}{1-R}A(T), \end{aligned}$$

where

$$\begin{aligned} A(T) &= \frac{1}{c_0^{1-R}}\mathbb{E}\left[\exp\left\{-(1-R)\left(\sigma W_T - \frac{1}{2}\sigma^2 T\right)\right\}\right] \\ &= \frac{\exp\left\{\frac{1}{2}\sigma^2 T(1-R)(2-R)\right\}}{c_0^{1-R}}. \end{aligned}$$

Duality gives that the optimal terminal wealth is

$$X_T^* = I\left(\lambda Z_T\right),$$

where $I(y) = y^{-\frac{1}{R}}$ and λ is such that

$$x_0 = \mathbb{E}[Z_T I\left(\lambda Z_T\right)] = \lambda^{-\frac{1}{R}}\mathbb{E}\left[Z_T^{\frac{R-1}{R}}\right] = \lambda^{-\frac{1}{R}}\exp\left\{\frac{1}{2}\mu^2 T\frac{1-R}{R^2}\right\}.$$

So

$$\begin{aligned} V_R^T\left(\frac{x_0}{c_0}\right) &= \frac{A(T)}{1-R}\mathbb{E}\left[(X_T^*)^{1-R}\right] \\ &= \frac{1}{1-R}\left(\frac{x_0}{c_0}\right)^{1-R}\exp\left\{\frac{1}{2}\mu^2 T\frac{1-R}{R} + \frac{1}{2}\sigma^2 T(1-R)(2-R)\right\}. \end{aligned}$$

Consider also the case where the claim is revealed at time γ where $0^+ \leq \gamma \leq T^-$. As before the claim has distribution

$$C = c_0 e^{\sigma\sqrt{T}G - \frac{1}{2}\sigma^2 T}.$$

So at time γ^-, just before the claim is revealed the optimal payoff is

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{1-R}\left(\frac{x_\gamma}{C}\right)^{1-R}\exp\left\{\frac{1}{2}\mu^2(T-\gamma)\frac{1-R}{R}\right\}\right] \\
&\quad = \frac{1}{1-R}\left(\frac{x_\gamma}{c_0}\right)^{1-R}\exp\left\{\frac{1}{2}\mu^2(T-\gamma)\frac{1-R}{R} + \frac{1}{2}\sigma^2 T(1-R)(2-R)\right\},
\end{aligned}$$

and so at time 0 the optimal expected payoff is

$$\begin{aligned}
V_R^\gamma\left(\frac{x_0}{c_0}\right) &= \frac{1}{1-R}\left(\frac{x_0}{c_0}\right)^{1-R}\exp\left\{\frac{1}{2}\mu^2(T-\gamma)\frac{1-R}{R} + \frac{1}{2}\sigma^2 T(1-R)(2-R)\right. \\
&\quad \left. + \frac{1}{2}\mu^2\gamma\frac{1-R}{R}\right\} \\
&= \frac{1}{1-R}\left(\frac{x_0}{c_0}\right)^{1-R}\exp\left\{\frac{1}{2}\mu^2 T\frac{1-R}{R} + \frac{1}{2}\sigma^2 T(1-R)(2-R)\right\},
\end{aligned}$$

which is the same as for the case with the claim revealed at maturity. The case where the claim is revealed before the outset is identical.

Let us compare this result with the cases of continuously revealed claims for constant relative risk aversion. We conjecture that the solution is of the form

$$V_R(u,0) = \frac{u^{1-R}}{1-R}g(T),$$

and so reduce our HJB equation (2.7) to

$$g(t)\left\{\left(1-\frac{1}{2}R\right)\sigma_t^2 + \frac{1}{2R}\left(\mu - \sigma_t\rho + R\sigma_t\rho\right)^2\right\} + \frac{\dot{g}(t)}{1-R} = 0,$$

which gives us

$$\begin{aligned}
V_R(u,0) = \frac{u^{1-R}}{1-R}\exp&\left\{\left(\frac{1}{2}(1-R)(2-R) + \left(\frac{1}{2}(1-R)(2-R) - \frac{1-R}{2R}\right)\rho^2\right)\right. \\
&\left.\times\int_0^T \sigma_s^2\,ds + \mu\rho\frac{(1-R)^2}{R}\int_0^T \sigma_s\,ds + \frac{1-R^2}{2R}T\right\}
\end{aligned}$$

Suppose we take $\int_0^T \sigma_s^2\,ds = (\sigma^*)^2 T$ and $\mu_t = \mu$ fixed. If we choose $\Sigma = \int_0^T \sigma_s\,ds$

to maximise the success probability, V, then we see that for $0 < R < 1$ we take σ_t such that

$$\sigma_t^2 = \begin{cases} (\sigma^*)^2 T\delta(t-\gamma) & \text{for } \mu\rho > 0 \\ (\sigma^*)^2 T & \text{for } \mu\rho < 0 \end{cases},$$

where $\delta(t)$ is a delta function with discontinuity at time 0. For $R > 1$ we take

$$\sigma_t^2 = \begin{cases} (\sigma^*)^2 T & \text{for } \mu\rho > 0 \\ (\sigma^*)^2 T\delta(t-\gamma) & \text{for } \mu\rho < 0 \end{cases}.$$

We notice that what time γ the spike in the volatility occurs does not affect the value function in this context.

We consider briefly why the form of the optimal volatility distribution depends on the sign of $(1-R)$. From (2.6) we have that the optimal policy is

$$\phi_t^* = \rho\sigma_t\left(1 - \frac{1}{R}\right) + \frac{\mu}{R},$$

so the extra stock holding required for hedging because of the correlation is

$$\tilde{\phi}_t = \rho\sigma_t \frac{R-1}{R}.$$

If we interpret $(\tilde{\phi}_t)$ as a policy then we find that the change in terminal wealth because of the correlation is

$$\int_0^T \tilde{\phi}_s \frac{dP_s}{P_s} = \mu\rho\frac{R-1}{R}\int_0^T \sigma_s\, ds + \text{ martingale term},$$

so we see that whether we want to maximise or minimise $\int_0^T \sigma_s\, ds$ depends on the sign of $\mu\rho(1-R)$.

We have the same success probability, when $\rho = 0$, as for the cases with instantaneously revealed claims. So, as in the quantile hedging case, the value function does not depend on the timing of volatility.

6.3 Concluding Remarks

We have investigated the problem facing an agent who seeks to meet a contingent claim, but who has insufficient funds to finance a replicating portfolio (in a complete market) or super-hedging portfolio (in an incomplete market). In particular we suppose the agent aims to maximise the probability of meeting the claim – the so-called quantile hedging problem.

Our attention has focused on markets where both the traded asset and the claim are geometric Brownian motions. There were several reasons for this including tractability, but also including realism and applicability. Moreover, such a framework allows us to consider the impact of correlation and of the timing of information.

In a complete market, and in a market with zero correlation and with traded assets which are martingales, we found explicit analytic solutions for the value function. We also showed that the general problem in which there is correlation between the asset and the claim can be reduced to the uncorrelated case.

The general problem with several traded assets can be reduced to a single traded asset. Further in our context other problems, such as minimising expected shortfall, can also be recast into a quantile hedging problem with modified parameters. Hence the simple structure which is the focus is a paradigm for a wider set of problems.

For the problem with a traded asset which is not a martingale, we used numerical methods to obtain a solution for the value function. We used a method of policy improvement to solve a sequence of linear partial differential equations. We were able to use the numerical solution to make comparisons with the solution when the traded asset is a martingale, and showed that the agent can take advantage of the drift to improve the probability of meeting the claim.

Of particular interest is a comparison of the optimal strategies. The optimal strategy in the case with (positive) drift is to hold a positive, state and time dependent, proportion of wealth in the risky asset. This proportion increases as the time to go decreases, or as the ratio of wealth to claim decreases. The case of zero drift can be thought of as a special case in which all the trading takes place at the last instant, and then only if the claim value exceeds current wealth.

A related problem is that where one may call in the claim if at any point before maturity one has wealth equal to or greater than the value of the claim. In the case of zero asset-drift we were able to represent the success probability in terms of the value of European options. By using barrier option methods we reduced an American problem to a European one. We also obtained the success probability numerically for the case where the asset is not a martingale. Surprisingly, the value improvement from having non-zero drift in the American problem is very similar to the corresponding value improvement in the European problem.

In this chapter we compared receiving information on the claim value continuously through time to the case where the value of the claim is revealed instantaneously. In the quantile hedging case, the agent does significantly worse in cases where no information is available during the trading period when compared with similar cases where the terminal claim value is revealed through time. It is an order of magnitude more useful to be able to observe the claim through time than it is to have precise knowledge of the drift μ and to be able to use the exact optimal strategy. However, the benefit from seeing the claim evolve through time and the benefit from having exact knowledge of the claim are of similar magnitude. The latter benefit is larger as we would expect.

In the next chapter we will look at something rather different. However, we will still be considering questions of timely information and our setting will still be Brownian.

Chapter 7

The Timely Approximation of Asian Options

An Asian option is a financial derivative the value of which depends on the average price of the underlying asset during the life of the option. In the context of a continuous time model the average is expressed as an integral.

Suppose we are only able to observe the value of the underlying a certain number of times but we wish to estimate the average from these observations. In this chapter we consider how best to choose the times and what functions of these time points make good estimates. Throughout the functions used to approximate the integrals will be linear. We use the $\mathbf{L}^2$-norm to judge the quality of our estimates.

7.1 Discussion of Related Work

This question was posed by Dr Martin Baxter of Nomura International. However, there is little discussion of ideas of this sort in the literature. Dufresne [12] obtains an expression for the density of the distribution of the integral of Brownian motion by considering the Laplace transform of its reciprocal. By considering exponential stopping times, Dufresne [11] obtains a relationship between the integral of geometric-Brownian motion with positive drift and the integral with corresponding negative drift.

Milevsky and Posner [41] show that the sum of infinitely many log-normal random variables has reciprocal gamma distribution, i.e. its inverse has gamma distribution. Donati-Martin, Ghomrasni and Yor [9] also consider Laplace transforms but of the integral of geometric Brownian motion. They obtain a closed form expression.

7.2 Brownian Assets

In this section we consider how best to choose observation times to approximate integrals of Brownian motion and its time changes. Although Brownian motion is a poor approximation to the movements of actual financial instruments, we hope that considering these more tractable cases will give us useful insights that will help with more realistic models.

We will use the notation

$$\int_0^T f(X_t)\,dt \rightsquigarrow g(\lambda_1,\ldots,\lambda_m;X_{\alpha_1},\ldots,X_{\alpha_n}),$$

to mean that we will approximate the integral on the left by the function on the right, which will depend on the random process, underlying the integral, but observed at a number of time points. The λ_i may be any real numbers, but clearly we require $\alpha_i \in [0,T]$. For concreteness, we will assume $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$.

The approximating function g will always be linear, for reasons of tractability, and so we will always have one parameter per time point, i.e. $m = n$.

We will denote by λ_i^* the optimising parameters of g for given fixed time points α_i, and similarly by α_i^* the optimising time points given fixed parameters. We will denote by $\lambda_i^\diamondsuit$ and $\alpha_i^\diamondsuit$ the jointly optimal parameters and time points.

We begin with the simplest case of approximating the integral of a standard Brownian motion by choosing one time point and one parameter in a linear function. That is we consider

$$\int_0^T B_t\,dt \rightsquigarrow \lambda B_\alpha,$$

in which we wish to choose $\lambda \in \mathbb{R}$ and $\alpha \in [0,T]$ to minimise the expected squared error

$$\begin{aligned}E(\lambda,\alpha;T) =&\mathbb{E}\left[\left(\int_0^T B_t\,dt - \lambda B_\alpha\right)^2\right]\\ =&\mathbb{E}\left[\int_0^T\left(\int_0^T B_s\,ds\right)B_t\,dt\right] - 2\lambda\mathbb{E}\left[B_\alpha\int_0^T B_t\,dt\right] + \lambda^2\mathbb{E}\left[B_\alpha^2\right],\end{aligned}$$

so using the covariance structure of Brownian motion we have

$$\begin{aligned}E(\lambda, \alpha; T) = &\int_0^T \int_0^T (s \wedge t)\, ds\, dt - 2\lambda \int_0^T (\alpha \wedge t)\, dt + \lambda^2 \alpha \\ =&\frac{1}{3}T^3 - 2\lambda \left(T\alpha - \frac{1}{2}\alpha^2 \right) + \lambda^2 \alpha \\ =&\frac{1}{3}T^3 - 2\lambda\alpha T + \alpha^2 \lambda + \lambda^2 \alpha.\end{aligned}$$

Minimising over the quadratic in λ gives

$$\lambda^* = T - \frac{1}{2}\alpha.$$

Minimising over the quadratic in α gives

$$\alpha^* = \begin{cases} T & \text{if } \lambda < 0 \\ T - \frac{1}{2}\lambda & \text{if } 0 \leq \lambda \leq 2T \\ 0 & \text{if } \lambda > 2T \end{cases}.$$

Taking λ very small, but positive, gives an optimal value of α^* very near T. If we are forced to take a very small multiple of the process at the observation time, we choose to make our observation time as late as possible in the hope that the process will have grown in magnitude.

Combining the expressions for λ^* and α^* we find the jointly minimising values of α and λ are $\alpha^\diamond = \frac{2}{3}T$ and $\lambda^\diamond = \frac{2}{3}T$.

So we have seen that with one time point we wait until two thirds of the time over which the integral is taken has passed then we approximate the integral by two thirds of the value of the process that we observe at that time.

A next step is to have multiple time points on which to base our approximation, that is to consider

$$\int_0^T B_t\, dt \rightsquigarrow \sum_{i=1}^n \lambda_i B_{\alpha_i},$$

We have

$$\begin{aligned}
&E(\lambda_1,\dots,\lambda_n,\alpha_1,\dots,\alpha_n;T)\\
&=\mathbb{E}\left[\left(\int_0^T B_t\,dt-\sum_{i=1}^n \lambda_i B_{\alpha_i}\right)^2\right]\\
&=\frac{1}{3}T^3-2\sum_{i=1}^n \lambda_i \mathbb{E}\left[B_{\alpha_i}\int_0^T B_t\,dt\right]+\sum_{i=1}^n \lambda_i^2 \mathbb{E}\left[B_{\alpha_i}^2\right]+2\sum_{i=1}^n\sum_{j=1}^{i-1}\lambda_i\lambda_j\mathbb{E}\left[B_{\alpha_i}B_{\alpha_j}\right].
\end{aligned}$$

Evaluating the remaining expectations we have

$$\begin{aligned}
&E(\lambda_1,\dots,\lambda_n,\alpha_1,\dots,\alpha_n;T)\\
&=\frac{1}{3}T^3-2T\sum_{i=1}^n\lambda_i\alpha_i+\sum_{i=1}^n\lambda_i\alpha_i^2+\sum_{i=1}^n\lambda_i^2\alpha_i+2\sum_{i=1}^n\sum_{j=1}^{i-1}\lambda_i\lambda_j\alpha_j.
\end{aligned}$$

Minimizing over each of the quadratics in λ_i gives

$$-2T\alpha_i+\alpha_i^2+2\lambda_i^*\alpha_i+2\sum_{j=1}^{i-1}\lambda_j^*\alpha_j+2\alpha_i\sum_{j=i+1}^{n}\lambda_j^*=0,$$

so

$$\alpha_i\lambda_i^*=\alpha_iT-\frac{1}{2}\alpha_i^2-\sum_{j=1}^{i-1}\lambda_j^*\alpha_j-\alpha_i\sum_{j=i+1}^{n}\lambda_j^*, \tag{7.1}$$

where the final sum is taken to be zero if $i+1>n$.

Lemma 7.1 *The system of n non-linear equations given by (7.1) is equivalent to the following system of n linear equations*

$$\begin{aligned}
\lambda_1^*&=\frac{1}{2}\alpha_2\\
\lambda_i^*&=\frac{1}{2}\left(\alpha_{i+1}-\alpha_{i-1}\right)\\
\lambda_n^*&=T-\frac{1}{2}\left(\alpha_n+\alpha_{n-1}\right),
\end{aligned}\tag{7.2}$$

providing the α_i are distinct.

Proof. We have, for $2 \leq i \leq n$,

$$\begin{aligned}\lambda_i^* \alpha_i &= T\alpha_i - \frac{1}{2}\alpha_i^2 - \alpha_i \sum_{j=i+1}^{n} \lambda_i^* - \sum_{j=1}^{i-1} \lambda_j^* \alpha_j \\ &= T\alpha_i - \frac{1}{2}\alpha_i^2 - \alpha_i \sum_{j=i+1}^{n} \lambda_j^* - \left(T\alpha_{i-1} - \frac{1}{2}\alpha_{i-1}^2 - \alpha_{i-1} \sum_{j=i}^{n} \lambda_j^* \right),\end{aligned}$$

since

$$\lambda_{i-1}^* \alpha_{i-1} = T\alpha_{i-1} - \frac{1}{2}\alpha_{i-1}^2 - \alpha_{i-1} \sum_{j=i}^{n} \lambda_j^* - \sum_{j=1}^{i-2} \lambda_j^* \alpha_j,$$

gives

$$\sum_{j=1}^{i-1} \lambda_j^* \alpha_j = T\alpha_{i-1} - \frac{1}{2}\alpha_{i-1}^2 - \alpha_{i-1} \sum_{j=i}^{n} \lambda_j^*.$$

Consequently,

$$\lambda_i^* \left(\alpha_i - \alpha_{i-1}\right) = T\left(\alpha_i - \alpha_{i-1}\right) - \frac{1}{2}\left(\alpha_i^2 - \alpha_{i-1}^2\right) - \left(\alpha_i - \alpha_{i-1}\right) \sum_{j=i+1}^{n} \lambda_j^*.$$

As the α_i are distinct we obtain

$$\lambda_i^* = T - \frac{1}{2}\left(\alpha_i + \alpha_{i-1}\right) - \sum_{j=i+1}^{n} \lambda_j^*,$$

which is the required result in the case $i = n$.

For $2 \leq i \leq n-1$, we have

$$\begin{aligned}\lambda_i^* &= T - \frac{1}{2}\left(\alpha_i + \alpha_{i-1}\right) - \lambda_{i+1}^* - \sum_{j=i+2}^{n} \lambda_j^* \\ &= T - \frac{1}{2}\left(\alpha_i + \alpha_{i-1}\right) - T + \frac{1}{2}\left(\alpha_{i+1} + \alpha_i\right) \\ &= \frac{1}{2}\left(\alpha_{i+1} - \alpha_{i-1}\right),\end{aligned}$$

as required. Finally,

$$\lambda_1^* = T - \frac{1}{2}\alpha_1 - \left(T - \frac{1}{2}\left(\alpha_2 + \alpha_1\right)\right) = \frac{1}{2}\alpha_2.$$

□

So given fixed observation times $\alpha_1, \alpha_2, \ldots \alpha_n$, the best approximation for the integral $\int_0^T B_t\, dt$ is

$$\begin{aligned}
&\frac{1}{2}\alpha_2 B_{\alpha_1} + \frac{1}{2}(\alpha_3 - \alpha_1) B_{\alpha_2} \ldots + \frac{1}{2}(\alpha_n - \alpha_{n-2}) B_{\alpha_{n-1}} \\
&\quad + \left(T - \frac{1}{2}(\alpha_n + \alpha_{n-1})\right) B_{\alpha_n} \\
&= \frac{1}{2}\alpha_1 (B_0 + B_{\alpha_1}) + \frac{1}{2}(\alpha_2 - \alpha_1)(B_{\alpha_1} + B_{\alpha_2}) + \ldots \\
&\quad + \frac{1}{2}(\alpha_n - \alpha_{n-1})(B_{\alpha_{n-1}} + B_{\alpha_n}) + (T - \alpha_n) B_{\alpha_n}.
\end{aligned}$$

The geometric interpretation of this is that, given the α_i, it is optimal to choose a trapezoidal approximation. This is shown in Figure 7-1.

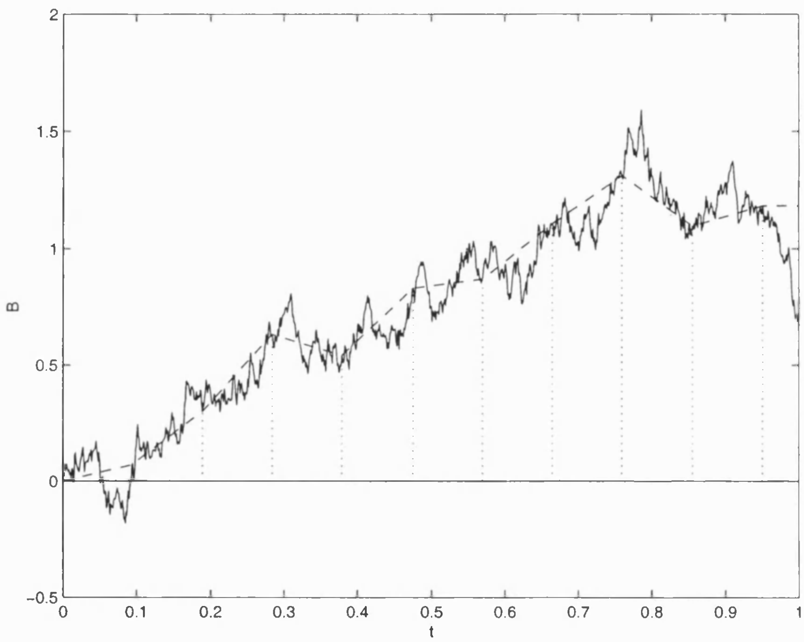

Figure 7-1: The Optimal Approximation of a Brownian integral is Trapezoidal

Next we want to find the α_i^*. Minimizing over each quadratic in α_i gives

$$-2T\lambda_i + 2\lambda_i\alpha_i^* + \lambda_i^2 + 2\lambda_i \sum_{j=i+1}^{n} \lambda_j = 0,$$

so

$$\alpha_i^* = T - \frac{1}{2}\lambda_i - \sum_{j=i+1}^{n} \lambda_j. \tag{7.3}$$

Proposition 7.1 *The unique joint solution to the* $2n$ *linear equations given by (7.2) and (7.3) is*

$$\lambda_i^{\diamond} = \frac{T}{n+\frac{1}{2}}, \quad \alpha_i^{\diamond} = \frac{iT}{n+\frac{1}{2}}.$$

Proof. Suppose $\alpha_1 = \eta$, then substituting (7.2) into (7.3),

$$\begin{aligned}
\eta &= T - \frac{1}{4}\alpha_2 - (\lambda_2 + \ldots + \lambda_n) \\
&= T - \frac{1}{4}\alpha_2 - \frac{1}{2}\Big\{(\alpha_3 - \eta) + (\alpha_4 - \alpha_2) + \ldots \\
&\qquad (\alpha_{n-1} - \alpha_{n-3}) + (\alpha_n - \alpha_{n-2})\Big\} - T + \frac{1}{2}(\alpha_n + \alpha_{n-1}) \\
&= -\frac{1}{4}\alpha_2 + \frac{1}{2}(\eta + \alpha_2),
\end{aligned}$$

so

$$\alpha_2 = 2\eta.$$

Similarly, for $2 \leq i \leq n-1$,

$$\alpha_i = -\frac{1}{4}(\alpha_{i+1} - \alpha_{i-1}) + \frac{1}{2}(\alpha_i + \alpha_{i+1}),$$

so

$$\alpha_{i+1} = 2\alpha_i - \alpha_{i-1}.$$

We conclude that

$$\alpha_i = i\eta,$$

for $1 \leq i \leq n$. However, we also have

$$\begin{aligned}
\alpha_n &= T - \frac{1}{2}\left(T - \frac{1}{2}(\alpha_n + \alpha_{n-1})\right) \\
&= \frac{1}{2}T + \frac{1}{4}\alpha_n + \frac{1}{4}\alpha_{n-1},
\end{aligned}$$

which gives

$$\alpha_n = \frac{2}{3}T + \frac{1}{3}\alpha_{n-1}.$$

From this we can obtain η since

$$n\eta = \frac{2}{3}T + \frac{1}{3}(n-1)\eta,$$

and so

$$\eta = \frac{T}{n+\frac{1}{2}},$$

as required. Finally we can obtain the λ_i,

$$\lambda_1 = \frac{1}{2}\alpha_2 = \frac{T}{n+\frac{1}{2}}$$
$$\lambda_i = \frac{1}{2}(\alpha_{i+1} - \alpha_{i-1}) = \frac{T}{n+\frac{1}{2}},$$

for $2 \leq i \leq n-1$, and

$$\begin{aligned}\lambda_n &= T\left(1 - \frac{1}{2\left(n+\frac{1}{2}\right)}(n+n-1)\right)\\ &= \frac{T}{n+\frac{1}{2}},\end{aligned}$$

as required. □

That is we obtain

$$\int_0^1 B_t\,dt \rightsquigarrow \sum_{i=1}^{n} \frac{T}{n+\frac{1}{2}} B_{\frac{iT}{n+\frac{1}{2}}}.$$

The time points chosen divide $[0,T]$ in the ratio two to two to ... to two to one. By doing this every point in the interval $[0,T]$ is at most $T/(2n+1)$ from an observed point, recalling that we know $B_0 = 0$. We notice that the $\lambda_i^\diamond$ are equal. We would expect this given the symmetry in the phrasing of the problem and the equal spacing of the $\alpha_i^\diamond$.

To summarise, we observe the process at intervals of $1/\left(n+\frac{1}{2}\right)$ of the integration interval and then approximate our integral by the sum of these values scaled by $1/\left(n+\frac{1}{2}\right)$.

In addition to choosing the appropriate location of our time points we are concerned with choosing an appropriate number of time points. We find using our earlier results that the minimal error is given by

$$\begin{aligned}
E(\lambda_1^\diamond,\dots,\lambda_n^\diamond,\alpha_1^\diamond,\dots,\alpha_n^\diamond;T) &= \frac{1}{3}T^3 - \frac{T^3}{\left(n+\frac{1}{2}\right)^3}\sum_{i=1}^n i^2 \\
&\quad + \left(\frac{2n+1}{\left(n+\frac{1}{2}\right)^3} - \frac{2}{\left(n+\frac{1}{2}\right)^2}\right)T^3\sum_{i=1}^n i \\
&= \frac{1}{3}T^3\left(1-\frac{n(n+1)}{3\left(n+\frac{1}{2}\right)^2}\right) \\
&= \frac{T^3}{12\left(n+\frac{1}{2}\right)^2}.
\end{aligned}$$

As an approximate heuristic, if we doubled the number of estimation points we would quarter the minimal error.

We can substantially increase the usefulness of our results by analysing the case of integrals of deterministic time changes of Brownian Motion. We saw in the previous case that there was no loss of generality in taking the multipliers in our linear approximation to be equal, $\lambda_1 = \dots = \lambda_n = \lambda$, and so we shall do the same here. That is we consider

$$\int_0^T B_{g(t)}\,dt \rightsquigarrow \lambda\sum_{i=1}^n B_{\alpha_i},$$

where g is continuous and strictly increasing with $g(0)=0$ and $g(T)=T$. Denoting $h=g^{-1}$ we have

$$\begin{aligned}
E(\lambda,\alpha_1,\dots,\alpha_n;T) &= \mathbb{E}\left[\left(\int_0^T B_{g(t)}\,dt - \lambda\sum_{i=1}^n B_{\alpha_i}\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^T\left(T-h(t)-\lambda\sum_{i=1}^n \mathbb{1}_{\{t\le\alpha_i\}}\right)dB_t\right)^2\right],
\end{aligned}$$

since

$$d\left(B_t(T-h(t))\right) = dB_t(T-h(t)) - d(h(t))B_t.$$

Now we can interchange the integration and expectation

$$\begin{aligned}
E(\lambda, \alpha_1, \ldots, \alpha_n; T) &= \int_0^T \left(T - h(t) - \lambda \sum_{i=1}^n \mathbb{1}_{\{t \le \alpha_i\}} \right)^2 dt \\
&= \int_0^T (T - h(t))^2 \, dt - 2\lambda \sum_{i=1}^n \int_0^{\alpha_i} (T - h(t)) \, dt \\
&\quad + \lambda^2 \sum_{i=1}^n \alpha_i (2n + 1 - 2i).
\end{aligned}$$

Minimising over λ we have

$$\lambda^* = \frac{\sum_{i=1}^n \int_0^{\alpha_i} (T - h(t)) \, dt}{\sum_{i=1}^n (2n + 1 - 2i)\alpha_i}, \tag{7.4}$$

and minimising over α_i we have

$$\lambda = \frac{T - h(\alpha_i^*)}{n + \frac{1}{2} - i},$$

that is

$$\alpha_i^* = g\left(T - \left(n + \frac{1}{2} - i \right) \lambda \right), \tag{7.5}$$

which we notice is similar in form to (7.3).

We can combine these two expressions to give the jointly minimising $\alpha_i^\diamond$ and $\lambda_i^\diamond$. In the case of $n = 1$ the optimising parameter values satisfy $\alpha^\diamond(T - h(\alpha^\diamond)) = \int_0^{\alpha^\diamond} (T - h(t)\, dt - \alpha^\diamond(T - h(\alpha^\diamond))$ and $\lambda^\diamond = 2(T - h(\alpha^\diamond))$. We can represent this first equation pictorially. It says that the two different shaded areas in Figure 7-2 are equal.

7.3 Geometric-Brownian Assets

A more realistic model for stock price movements than Brownian motion is that of geometric Brownian motion. In addition to always being positive, geometric Brownian motion has the property that incremental changes are proportional to current magnitude. However, this extra realism comes at a cost in reduced tractability. This is particular so in the case of Asian options as they are defined in an additive manner whereas geometric Brownian motion possesses useful multiplicative properties.

We begin by considering the case where the geometric Brownian motion is a martingale with unit volatility and we have only one time point in our approximation, that

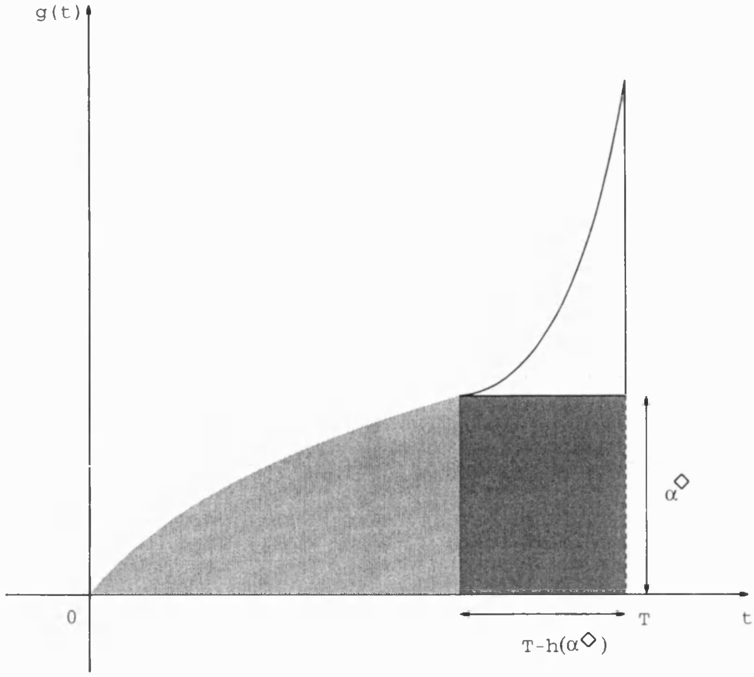

Figure 7-2: Depiction of $\alpha^{\diamondsuit}(T - h(\alpha^{\diamondsuit})) = \int_0^{\alpha^{\diamondsuit}} (T - h(t))\, dt - \alpha^{\diamondsuit}(T - h(\alpha^{\diamondsuit}))$

is we consider

$$\int_0^1 e^{B_t - \frac{1}{2}t}\, dt \rightsquigarrow \lambda e^{B_\alpha - \frac{1}{2}\alpha}.$$

Now we have

$$\begin{aligned} E(\lambda, \alpha; T) &= \mathbb{E}\left[\left(\int_0^T e^{B_t - \frac{1}{2}t}\, dt - \lambda e^{B_\alpha - \frac{1}{2}\alpha}\right)^2\right] \\ &= \mathbb{E}\left[\left(\int_0^T e^{B_t - \frac{1}{2}t}\, dt\right)^2\right] - 2\lambda \int_0^T \mathbb{E}\left[e^{B_\alpha - \frac{1}{2}\alpha} e^{B_t - \frac{1}{2}t}\right] dt \\ &\quad + \lambda^2 \mathbb{E}\left[\left(e^{B_\alpha - \frac{1}{2}\alpha}\right)^2\right]. \end{aligned}$$

Evaluating the expectation

$$\begin{aligned} E(\lambda, \alpha; T) &= \mathbb{E}\left[\left(\int_0^T e^{B_t - \frac{1}{2}t}\, dt\right)^2\right] - 2\lambda\left(\int_0^\alpha e^t\, dt + \int_\alpha^T e^\alpha\, dt\right) + \lambda^2 e^\alpha \\ &= \mathbb{E}\left[\left(\int_0^T e^{B_t - \frac{1}{2}t}\, dt\right)^2\right] - 2\lambda\left(e^\alpha(T + 1 - \alpha) - 1\right) + \lambda^2 e^\alpha. \end{aligned}$$

Using that this is a quadratic in λ with positive λ^2 coefficient, we have that

$$\lambda^* = T + 1 - \alpha - e^{-\alpha} \tag{7.6}$$

Substituting this value and minimising $E(\lambda^*, \alpha; T)$ over α gives

$$e^{-\alpha^{\diamondsuit}} = \alpha^{\diamondsuit} + 1 - T. \tag{7.7}$$

So, for example, when T=1, $\alpha^{\diamondsuit} \approx 0.567$ and so $\lambda^{\diamondsuit} \approx 0.866$.

We continue with one time point but drop the martingale condition and unit volatility of the geometric Brownian motion, so we want

$$\int_0^T e^{\sigma B_t + \mu t}\, dt \rightsquigarrow \lambda e^{\sigma B_\alpha + \mu\alpha}$$

we have

$$\begin{aligned} E(\lambda, \alpha; T) = \mathbb{E}\left[\left(\int_0^T e^{\sigma B_t + \mu t}\, dt\right)^2\right] - 2\lambda \int_0^T \mathbb{E}\left[e^{\sigma B_t + \mu t} e^{\sigma B_\alpha + \mu\alpha}\right] dt \\ + \lambda^2 \mathbb{E}\left[\left(e^{\sigma B_\alpha + \mu\alpha}\right)^2\right]. \end{aligned}$$

For convenience of exposition we denote

$$\tilde{E}(\lambda, \alpha; T) = E(\lambda, \alpha; T) - \mathbb{E}\left[\left(\int_0^T e^{\sigma B_t + \mu t}\, dt\right)^2\right].$$

So

$$\begin{aligned} \tilde{E}(\lambda, \alpha; T) = -2\lambda\left(e^{(\mu + \frac{1}{2}\sigma^2)\alpha}\int_0^\alpha e^{(\mu + \frac{3}{2}\sigma^2)t}\, dt + e^{(\mu + \frac{3}{2}\sigma^2)\alpha}\int_\alpha^T e^{(\mu + \frac{1}{2}\sigma^2)t}\, dt\right) \\ + \lambda^2 e^{2(\mu + \sigma^2)\alpha}. \end{aligned}$$

Performing the integration gives, when $\mu \neq -\frac{1}{2}\sigma^2,\ -\frac{3}{2}\sigma^2$,

$$\begin{aligned} \tilde{E}(\lambda, \alpha; T) = -2\lambda\Bigg(\frac{e^{(\mu + \frac{1}{2}\sigma^2)\alpha}}{\mu + \frac{3}{2}\sigma^2}\left(e^{(\mu + \frac{3}{2}\sigma^2)\alpha} - 1\right) \\ + \frac{e^{(\mu + \frac{3}{2}\sigma^2)\alpha}}{\mu + \frac{1}{2}\sigma^2}\left(e^{(\mu + \frac{1}{2}\sigma^2)T} - e^{(\mu + \frac{1}{2}\sigma^2)\alpha}\right)\Bigg) + \lambda^2 e^{2(\mu + \sigma^2)\alpha}, \end{aligned} \tag{7.8}$$

which rearranges to

$$\tilde{E}(\lambda,\alpha;T) = -2\lambda\left(\frac{e^{2(\mu+\sigma^2)\alpha} - e^{\left(\mu+\frac{1}{2}\sigma^2\right)\alpha}}{\mu+\frac{3}{2}\sigma^2} + \frac{e^{\left(\mu+\frac{1}{2}\sigma^2\right)T+\left(\mu+\frac{3}{2}\sigma^2\right)\alpha} - e^{2(\mu+\sigma^2)\alpha}}{\mu+\frac{1}{2}\sigma^2}\right) + \lambda^2 e^{2(\mu+\sigma^2)\alpha}.$$

Now minimising over λ gives

$$\tilde{E}(\lambda^*,\alpha;T) = -\left(\frac{e^{\left(\mu+\sigma^2\right)\alpha} - e^{-\frac{1}{2}\sigma^2\alpha}}{\mu+\frac{3}{2}\sigma^2} + \frac{e^{\left(\mu+\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\sigma^2\alpha} - e^{\left(\mu+\sigma^2\right)\alpha}}{\mu+\frac{1}{2}\sigma^2}\right)^2$$

To minimise this over α, and so obtain $\alpha^\diamondsuit$, we maximise

$$f_{\mu,\sigma}(\alpha) = \frac{e^{\left(\mu+\sigma^2\right)\alpha} - e^{-\frac{1}{2}\sigma^2\alpha}}{\mu+\frac{3}{2}\sigma^2} + \frac{e^{\left(\mu+\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\sigma^2\alpha} - e^{\left(\mu+\sigma^2\right)\alpha}}{\mu+\frac{1}{2}\sigma^2}.$$

We have

$$f'_{\mu,\sigma}(\alpha) = \frac{\left(\mu+\sigma^2\right)e^{\left(\mu+\sigma^2\right)\alpha} + \frac{1}{2}\sigma^2 e^{-\frac{1}{2}\sigma^2\alpha}}{\mu+\frac{3}{2}\sigma^2} + \frac{\frac{1}{2}\sigma^2 e^{\left(\mu+\frac{1}{2}\sigma^2\right)T+\frac{1}{2}\sigma^2\alpha} - \left(\mu+\sigma^2\right)e^{\left(\mu+\sigma^2\right)\alpha}}{\mu+\frac{1}{2}\sigma^2},$$

which can be rewritten as

$$f'_{\mu,\sigma}(\alpha) = \frac{\sigma^2 e^{\left(\mu+\sigma^2\right)\alpha}}{2\left(\mu+\frac{1}{2}\sigma^2\right)\left(\mu+\frac{1}{2}\sigma^2\right)}\left\{(\mu+\frac{1}{2}\sigma^2)e^{-(\mu+\frac{3}{2}\sigma^2)\alpha} + (\mu+\frac{3}{2}\sigma^2)e^{(\mu+\frac{1}{2}\sigma^2)(T-\alpha)} - 2(\mu+\sigma^2)\right\}.$$

Hence the required $\alpha^\diamondsuit$ satisfies

$$(\mu+\frac{1}{2}\sigma^2)e^{-(\mu+\frac{3}{2}\sigma^2)\alpha^\diamondsuit} + (\mu+\frac{3}{2}\sigma^2)e^{(\mu+\frac{1}{2}\sigma^2)(T-\alpha^\diamondsuit)} = 2(\mu+\sigma^2). \tag{7.9}$$

Further

$$f''_{\mu,\sigma}(\alpha) = \frac{\sigma^2 e^{\mu+\sigma^2}}{2\left(\mu+\frac{1}{2}\sigma^2\right)\left(\mu+\frac{3}{2}\sigma^2\right)}\left\{ \left(\mu+\sigma^2\right)\left((\mu+\frac{1}{2}\sigma^2)e^{-(\mu+\frac{3}{2}\sigma^2)\alpha} \right.\right.$$
$$\left. + (\mu+\frac{3}{2}\sigma^2)e^{(\mu+\frac{1}{2}\sigma^2)(T-\alpha)} - 2(\mu+\sigma^2)\right)$$
$$\left. - \left(\mu+\frac{1}{2}\sigma^2\right)\left(\mu+\frac{3}{2}\sigma^2\right)\left(e^{-(\mu+\frac{3}{2}\sigma^2)\alpha} + e^{(\mu+\frac{1}{2}\sigma^2)(T-\alpha)}\right)\right\},$$

so

$$f''_{\mu,\sigma}(\alpha^\diamondsuit) = -\frac{1}{2}\sigma^2 e^{(\mu+\sigma^2)\alpha^\diamondsuit}\left(e^{-(\mu+\frac{3}{2}\sigma^2)\alpha^\diamondsuit} + e^{(\mu+\frac{1}{2}\sigma^2)(T-\alpha^\diamondsuit)}\right),$$

which is always positive. That is, any solution to (7.9) is a local minimum.

An immediate consequence of (7.9) is the symmetry

$$\alpha^\diamondsuit(\mu,\sigma) = T - \alpha^\diamondsuit(-\mu-2\sigma^2,\sigma). \tag{7.10}$$

Henderson and Wojakowski [22] consider a symmetry between floating-strike and fixed-strike Asian options. We follow their approach of using a change of numéraire.

We have

$$\begin{aligned} E(\lambda,\alpha;T) &= \mathbb{E}\left[\left(\int_0^T e^{\sigma B_t+\mu t}\,dt - \lambda e^{\sigma B_\alpha+\mu\alpha}\right)^2\right] \\ &= \mathbb{E}\left[e^{2\sigma B_T+2\mu T}\left(\int_0^T e^{\sigma(B_t-B_T)+\mu(t-T)}\,dt - \lambda e^{\sigma(B_\alpha-B_T)+\mu(\alpha-T)}\right)^2\right] \\ &= e^{2(\mu+\sigma^2)T}\hat{\mathbb{E}}\left[\left(\int_0^T e^{\sigma(B_t-B_T)+\mu(t-T)}\,dt - \lambda e^{\sigma(B_\alpha-B_T)+\mu(\alpha-T)}\right)^2\right], \end{aligned}$$

where

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{2\sigma B_T-2\sigma^2 T}.$$

Under $\hat{\mathbb{P}}$, $\hat{B}_t = B_t - 2\sigma t$ is a Brownian motion and so we rewrite $E(\lambda,\alpha;T)$ thus

$$\begin{aligned} E(\lambda,\alpha;T) = e^{2(\mu+\sigma^2)T}\hat{\mathbb{E}}\Bigg[\Bigg(\int_0^T & e^{\sigma\left(\hat{B}_t-\hat{B}_T\right)+\left(\mu+2\sigma^2\right)(t-T)}\,dt \\ & - \lambda e^{\sigma\left(\hat{B}_\alpha-\hat{B}_T\right)+\left(\mu+2\sigma^2\right)(\alpha-T)}\Bigg)^2\Bigg]. \end{aligned}$$

Furthermore, if we define $\check{B}_t = \hat{B}_{T-t} - \hat{B}_T$ then

$$E(\lambda, \alpha; T) = e^{2(\mu+\sigma^2)T}\hat{\mathbb{E}}\left[\left(\int_0^T e^{\sigma \check{B}_{T-t}-(\mu+2\sigma^2)(T-t)}\,dt - \lambda e^{\sigma \check{B}_{T-\alpha}-(\mu+2\sigma^2)(T-\alpha)}\right)^2\right].$$

If we reverse time by taking $s = T - t$ and $\beta = T - \alpha$ we have

$$E(\lambda, \alpha; T) = e^{2(\mu+\sigma^2)T}\hat{\mathbb{E}}\left[\left(\int_0^T e^{\sigma \check{B}_s-(\mu+2\sigma^2)s}\,ds - \lambda e^{\sigma \check{B}_\beta-(\mu+2\sigma^2)\beta}\right)^2\right].$$

However, $\check{B}_t$ is a Brownian motion under $\hat{\mathbb{P}}$ and so the expectation on the right becomes $E(\lambda, \alpha; T)$ under the transformation

$$\mu \mapsto -\mu - 2\sigma^2, \quad \alpha \mapsto T - \alpha,$$

which explains the symmetry (7.10).

In performing the integration to obtain (7.8) we assumed $\mu \neq -\frac{1}{2}\sigma^2, \frac{3}{2}\sigma^2$. Let us now consider the case $\mu = -\frac{1}{2}\sigma^2$. We have

$$\tilde{E}(\lambda, \alpha; T) = -2\lambda\left(\int_0^\alpha e^{\sigma^2 t}\,dt + e^{\sigma^2\alpha}\int_\alpha^T dt\right) + \lambda^2 e^{\sigma^2\alpha}.$$

Integrating, then minimising over λ gives

$$\tilde{E}(\lambda^*, \alpha; T) = -\left(\frac{e^{\frac{1}{2}\sigma^2\alpha} - e^{-\frac{1}{2}\sigma^2\alpha}}{\sigma^2} + (T-\alpha)e^{\frac{1}{2}\sigma^2\alpha}\right)^2$$

Minimising over α and simplifying gives

$$\sigma^2(T - \alpha^\diamond) - 1 + e^{-\sigma^2\alpha^\diamond} = 0. \tag{7.11}$$

Taking $\sigma = 1$ recovers (7.7).

Heuristically we can obtain this directly from (7.9). Writing $\delta = \mu + \frac{1}{2}\sigma^2$ we have

$$\delta e^{-(\delta+\sigma^2)\alpha^\diamond} + \left(\delta + \sigma^2\right) e^{\delta(T-\alpha^\diamond)} - 2\delta - \sigma^2 = 0.$$

Now for δ small we can neglect terms of order δ^2 and higher, and so the left hand side is approximately

$$\delta(1 - \alpha^\diamond\delta)e^{-\sigma^2\alpha^\diamond} + \left(\delta + \sigma^2\right)\left(1 + \delta(T - \alpha^\diamond)\right) - 2\delta - \sigma^2,$$

which is approximately

$$\delta + \sigma^2 + \delta\sigma^2(T - \alpha^\diamondsuit) + \delta e^{-\sigma^2\alpha^\diamondsuit} - 2\delta - \sigma^2.$$

Setting this equal to zero we obtain (7.11).

Similarly, if we take $\mu = -\frac{3}{2}\sigma^2$ we obtain

$$e^{-\sigma^2(T-\alpha^\diamondsuit)} = 1 - \sigma^2\alpha^\diamondsuit,$$

which in the $T = 1$, $\sigma = 1$ case gives $\alpha^\diamondsuit \approx 0.433$.

Proposition 7.2 *For all μ, σ such that $\mu \neq -\frac{1}{2}\sigma^2, -\frac{3}{2}\sigma^2$, the equation (7.9) has exactly one solution in $[0,T]$.*

Proof. Let

$$g_{\mu,\sigma}(\alpha) = (\mu + \frac{1}{2}\sigma^2)\left(e^{-(\mu+\frac{3}{2}\sigma^2)\alpha} - 1\right) + \left(\mu + \frac{3}{2}\sigma^2\right)\left(e^{\left(\mu+\frac{1}{2}\sigma^2\right)(T-\alpha)} - 1\right).$$

Since

$$g'_{\mu,\sigma}(\alpha) = -\left(\mu + \frac{1}{2}\sigma^2\right)\left(\mu + \frac{3}{2}\sigma^2\right)\left(e^{-(\mu+\frac{3}{2}\sigma^2)\alpha} + e^{\left(\mu+\frac{1}{2}\sigma^2\right)(T-\alpha)}\right),$$

g is always monotone and so (7.9) has at most one solution

Further

$$g_{\mu,\sigma}(0) = \left(\mu + \frac{3}{2}\sigma^2\right)\left(e^{\left(\mu+\frac{1}{2}\sigma^2\right)T} - 1\right)\begin{cases} > 0 & \text{if } \mu > -\frac{1}{2}\sigma^2 \\ < 0 & \text{if } -\frac{3}{2}\sigma^2 < \mu < -\frac{1}{2}\sigma^2 \\ > 0 & \text{if } \mu < -\frac{3}{2}\sigma^2 \end{cases},$$

and

$$g_{\mu,\sigma}(T) = \left(\mu + \frac{1}{2}\sigma^2\right)\left(e^{-(\mu+\frac{3}{2}\sigma^2)T} - 1\right)\begin{cases} < 0 & \text{if } \mu > -\frac{1}{2}\sigma^2 \\ > 0 & \text{if } -\frac{3}{2}\sigma^2 < \mu < -\frac{1}{2}\sigma^2 \\ < 0 & \text{if } \mu > -\frac{3}{2}\sigma^2 \end{cases}.$$

So by the intermediate value theorem there is always a solution and it lies in $[0,T]$. $\square$

For $\mu = -\frac{1}{2}\sigma^2$ and $\mu = -\frac{3}{2}\sigma^2$ the equation (7.9) is trivially satisfied for all α.

Figure 7-3 shows a plot of $\tilde{E}(\lambda^\diamondsuit, \alpha; 1)$ in the case $\sigma^2 = 1$. We see the unique minimsing $\alpha^\diamondsuit$ demonstrated by Proposition 7.2.

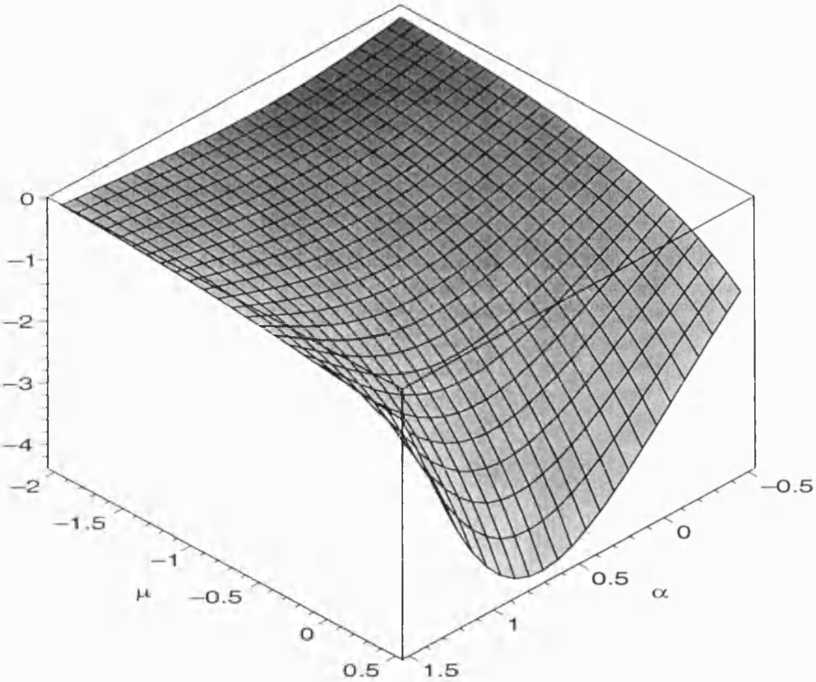

Figure 7-3: $\bar{E}(\lambda^{\diamond}, \alpha; 1)$ in the case $\int_0^T e^{\sigma B_t + \mu t}\, dt \rightsquigarrow \lambda e^{\sigma B_\alpha + \mu\alpha}$ with $\sigma^2 = 1$

Let us consider how $\alpha^{\diamond}$ varies with μ. Differentiating (7.9) with respect to μ gives us

$$\frac{\partial \alpha^{\diamond}}{\partial \mu} = \frac{e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^{\diamond}\right)} + e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}} - 2}{\left(\mu+\frac{3}{2}\sigma^2\right)\left(\mu+\frac{1}{2}\sigma^2\right)\left(e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^{\diamond}\right)} + e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}}\right)}.$$

Clearly

$$e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^{\diamond}\right)} + e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}} > 0.$$

We can rewrite the numerator, using (7.9), like this

$$e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^{\diamond}\right)} + e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}} - 2 = \frac{\sigma^2}{\mu+\frac{1}{2}\sigma^2}\left(1 - e^{\left(T-\alpha^{\diamond}\right)\left(\mu+\frac{1}{2}\sigma^2\right)}\right),$$

and like this

$$e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^{\diamond}\right)} + e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}} - 2 = -\frac{\sigma^2}{\mu+\frac{3}{2}\sigma^2}\left(1 - e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^{\diamond}}\right).$$

Hence we have that $\partial\alpha^{\diamond}/\partial\mu \geq 0$ for all real μ. As we increase the drift we increase the magnitude of the integral. We also increase the magnitude of the approximating function, especially at later time points. Consequently, it makes sense that as we increase the drift we prefer observation times to be later on in the integration interval.

Differentiating again and simplifying we obtain

$$\frac{\partial^2 \alpha^\diamond}{\partial \mu^2} = \frac{-2\left(\left(\mu+\frac{1}{2}\sigma^2\right)e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^\diamond\right)}+\left(\mu+\frac{3}{2}\sigma^2\right)e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^\diamond}\right)}{\left(\mu+\frac{3}{2}\sigma^2\right)\left(\mu+\frac{1}{2}\sigma^2\right)\left(e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^\diamond\right)}+e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^\diamond}\right)^2},$$

so for $\mu > -\frac{1}{2}\sigma^2$, $\alpha^\diamond$ is a concave function of μ and for $\mu < -\frac{3}{2}\sigma^2$, $\alpha^\diamond$ is a convex function of μ.

Turning briefly to the asymptotic behaviour of $\alpha^\diamond$ as a function of μ, we note that (7.9) can be rewritten as

$$\left(1+\frac{3\sigma^2}{2\mu}\right)e^{\left(\mu+\frac{1}{2}\right)\left(T-\alpha^\diamond\right)}+\left(1+\frac{\sigma^2}{2\mu}\right)e^{-\left(\mu+\frac{3}{2}\sigma^2\right)\alpha^\diamond}=2\left(1+\frac{\sigma^2}{\mu}\right).$$

As $\mu \to \infty$ the right handside is finite whereas the left hand side is only finite if $\alpha^\diamond \to T$. We expect this behaviour since for large drifts both the integral and the integrand grow rapidly and so we will best approximate the integral by choosing a later point for observing the integrand. Similarly we find that $\alpha^\diamond \to 0$ as $\mu \to -\infty$, which we expected given the symmetry (7.10).

From optimising the quadratic in $E(\lambda, \alpha; T)$, we have that the optimising λ is given by

$$\lambda^* = \frac{1-e^{-\alpha\left(\mu+\frac{3}{2}\sigma^2\right)}}{\mu+\frac{3}{2}\sigma^2} - \frac{1-e^{(T-\alpha)\left(\mu+\frac{1}{2}\sigma^2\right)}}{\mu+\frac{1}{2}\sigma^2}.$$

Rearranging (7.9) and substituting we have

$$\begin{aligned}\lambda^\diamond =& \frac{1}{\mu+\frac{3}{2}\sigma^2}+\frac{1}{\mu+\frac{1}{2}\sigma^2}\left(2e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^\diamond\right)}-\frac{2\left(\mu+\sigma^2\right)}{\mu+\frac{3}{2}\sigma^2}-1\right)\\ =& \frac{2}{\mu+\frac{1}{2}\sigma^2}\left(e^{\left(\mu+\frac{1}{2}\sigma^2\right)\left(T-\alpha^\diamond\right)}-1\right).\end{aligned}$$

We notice that since $\alpha^\diamond < T$ we have $\lambda^\diamond > 0$.

We make our model slightly more sophisticated by adding another approximation point but to keep things as simple as possible we have this fixed at the value 1. We also keep the number of parameters at two by taking a convex combination. That is we take

$$\int_0^1 e^{\sigma B_t+\mu t}\,dt \rightsquigarrow (1-\lambda)+\lambda e^{\sigma B_\alpha+\mu\alpha},$$

with the added restriction that $0 \leq \lambda \leq 1$. We find that

$$
\begin{aligned}
\tilde{E}(\lambda, \alpha; T) = & -2\lambda\left(\frac{e^{2(\mu+\sigma^2)\alpha} - e^{(\mu+\frac{1}{2}\alpha^2)\alpha}}{\mu+\frac{3}{2}\sigma^2}\right. \\
& + \left.\frac{e^{(\mu+\frac{1}{2}\sigma^2)T+(\mu+\frac{3}{2}\sigma^2)\alpha} - e^{2(\mu+\sigma^2)\alpha} - e^{(\mu+\frac{1}{2}\sigma^2)T} + 1}{\mu+\frac{1}{2}\sigma^2} - 1 + e^{(\mu\frac{1}{2}\sigma^2)\alpha}\right) \\
& + \lambda^2\left(1 - 2e^{(\mu+\frac{1}{2}\sigma^2)\alpha} + e^{2(\mu+\frac{1}{2}\sigma^2)\alpha}\right).
\end{aligned}
$$

As this is quadratic in λ with positive λ^2 coefficient we have that

$$
\begin{aligned}
\tilde{E}(\lambda^*, \alpha; T) = & -\left(\frac{e^{2\alpha(\mu+\sigma^2)} - e^{\alpha(\mu+\frac{1}{2}\alpha^2)}}{\mu+\frac{3}{2}\sigma^2}\right. \\
& + \frac{e^{T(\mu+\frac{1}{2}\sigma^2)+\alpha(\mu+\frac{3}{2}\sigma^2)} - e^{2\alpha(\mu+\sigma^2)} - e^{T(\mu+\frac{1}{2}\sigma^2)} + 1}{\mu+\frac{1}{2}\sigma^2} \\
& \left. - 1 + e^{\alpha(\mu\frac{1}{2}\sigma^2)}\right)^2 \times \frac{1}{1 - 2e^{\alpha(\mu+\frac{1}{2}\sigma^2)} + e^{2\alpha(\mu+\frac{1}{2}\sigma^2)}}.
\end{aligned}
$$

Figure 7-4 shows a plot of $\tilde{E}(\lambda^*, \alpha; 1)$, in the case where $\sigma^2 = 1$ and $T = 1$.

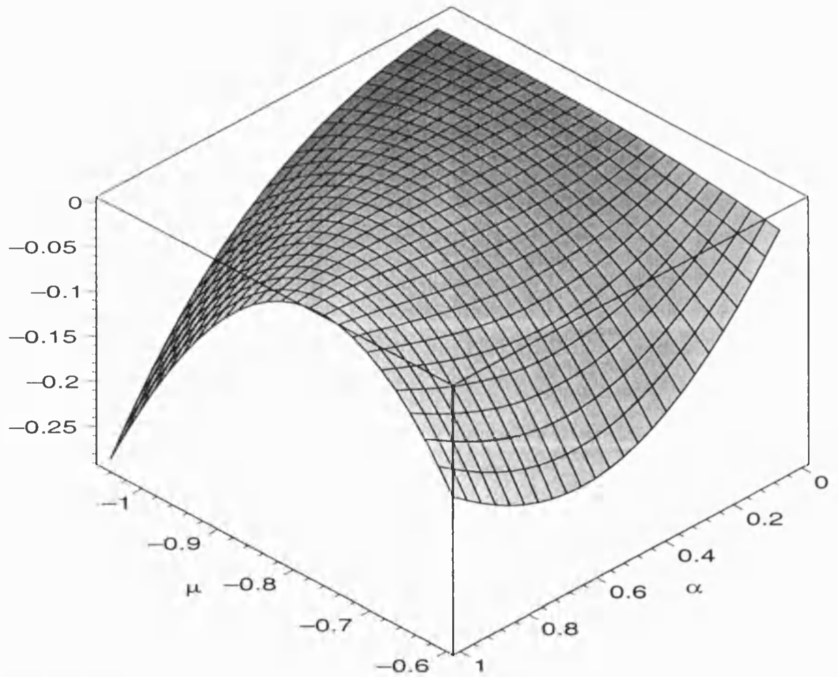

Figure 7-4: $\tilde{E}(\lambda^*, \alpha; 1)$ in the case $\int_0^T e^{\sigma B_t+\mu t}\, dt \rightsquigarrow (1-\lambda) + \lambda e^{\sigma B_\alpha+\mu\alpha}$ with $\sigma^2 = 1$

Suppose we define by $\tilde{\mu}$ the μ that maximises $E(\lambda^*, 1; T)$ as a function of μ. Given this we can describe three different type of behaviour in this model. For $\mu < -\sigma^2$, there are no local minima and the error is minimised by $\alpha^\diamondsuit = T$. For $-\sigma^2 < \mu < \tilde{\mu}$ we have a unique local minimum in $\tilde{\alpha}$ but the error is minimised by $\alpha^\diamondsuit = T$. Finally, for $\mu > \tilde{\mu}$ the error is minimised by the local minimum, that is $\alpha^\diamondsuit = \tilde{\alpha}$.

For sufficiently negative drift we find that it is best to take our observation of the geometric Brownian motion at the end of the integration interval. When the drift is substantially negative the integrand becomes small rapidly, giving a small integral. Consequently we best approximate the integral by observing the integrand at the end of the integration interval.

Unfortunately we are not able to extend this analysis to the case with two freely varying parameters, that is

$$\int_0^T e^{\sigma B_t + \mu t}\, dt \rightsquigarrow \lambda_1 + \lambda_2 e^{\sigma B_\alpha + \mu\alpha}.$$

It may be that an entirely numerical approach would be suitable for this problem or even the more general problem with n parameters and n observation times

$$\int_0^T e^{\sigma B_t + \mu t}\, dt \rightsquigarrow \sum_{i=1}^{n} \lambda_i e^{\sigma B_{\alpha_i} + \mu\alpha_i}.$$

It would also be of interest to investigate the problem with a time change of geometric Brownian motion

$$\int_0^T e^{\sigma B_{g(t)} + \mu g(t)}\, dt \rightsquigarrow \lambda e^{\sigma B_\alpha + \mu\alpha}.$$

Again it will probably be necessary to use a numerical approach here.

Chapter 8

A Stochastic Volatility Model

We now turn to a slightly different problem of optimal trading. As before, take

$$\frac{dX_t^\phi}{X_t^\phi} = \phi_t \frac{dP_t}{P_t}. \tag{8.1}$$

Interest rates will not feature in our model. However, as we could have considered discounted prices this will not result in a substantial loss of generality.

Now take

$$\frac{dP_t}{P_t} = Y_t \left(dW_t^1 + \lambda_t \, dt\right), \tag{8.2}$$

where the volatility Y is given by

$$dY_t = a\left(Y_t\right) \, dW_t^2 + b\left(Y_t\right) \, dt. \tag{8.3}$$

We use this slightly unusual parameterisation as we shall find later that λ plays a central role in our calculations and results. This quantity is often called the *Sharpe ratio.* Hodges [26] and Dowd [10] give interesting discussions of its interpretation. We let the Brownian motion driving the volatility be partially correlated with that directly driving the stock price,

$$dW_t^1 dW_t^2 = \rho dt.$$

We shall have constant relative risk aversion, as we did in Section 6.2. That is instead of maximising the chance of meeting a claim we shall maximise the utility

$$U(x) = \begin{cases} \frac{x^{1-R}}{1-R} & \text{if } R \neq 1 \\ \log x & \text{if } R = 1 \end{cases}, \tag{8.4}$$

for some $R > 0$. This is the problem of Merton [40], but we shall solve it with stochastic volatility.

We tackle this problem without the need for an assumption that the processes involved are Markovian. In addition to obtaining the value function and the optimal policy we are able to obtain the pricing measure.

8.1 Discussion of Related Work

In the early 1960s there had been some attempt to fit log daily stock returns to distributions more complicated than geometric Brownian motion. Mandelbrot [38] is an example of such empirical work. Around the late 1980s substantial interest began to be paid to pricing models that used stochastic volatility. Hobson [24] gives a good review of much of the literature.

Mostly, in the literature the drift of the process P is constant, that is $\lambda(y) = \frac{\mu}{y}$, for some constant μ. Hull and White [28] have a geometric-Brownian volatility. That is they have

$$a(y) = \alpha y, \quad b(y) = \beta y.$$

They obtain an analytic formula for the price of a European call in the case when there is no correlation between stock price and volatility. Hull and White [29] obtains series expansions in the case of non-zero correlation. Wiggins [52] applies numerical techniques in the case of non-zero correlation. Frey [18] characterises equivalent local martingale measures, that is equivalent probability measures in which the traded asset is a local martingale. He uses this to obtain bounds for option prices. He evaluates these bounds for the model of Hull and White [28].

Scott [48] applies Monte Carlo techniques in the case where the volatility is an Ornstein-Uhlenbeck process. That is

$$a(y) = \alpha, \quad b(y) = \beta - \gamma y.$$

Stein and Stein [50] also have volatility as an Ornstein-Uhlenbeck process but obtain explicit solutions in the case with no correlation.

Scott [48] also applies such techniques to the case where volatility is a geometric-Ornstein-Uhlenbeck process. In this case

$$a(y) = \alpha y, \quad b(y) = \beta y - \gamma y \log y.$$

We shall find an explicit solution for this model.

Hull and White [29] propose a model with

$$a(y) = \alpha, \quad b(y) = \frac{\beta}{y} - \gamma y.$$

That is the variance (i.e. the squared volatility) is a squared Bessel process, of the sort used by Cox, Ingersoll and Ross [8]. Heston [23] tackles this model numerically. Ball and Roma [3] apply Fourier methods to this model and that of Stein and Stein [50]

Johnson and Shanno [30] have a model where the stock price and the volatility are both constant elasticity of variance (CEV) processes. That is they have

$$dP_t = Y_t P_t^{\alpha} dW_t^1 + P_t dt,$$

and

$$dY_t = a\left(Y_t\right) Y_t^{\beta} dW_t^2 + b\left(Y_t\right) dt,$$

where $\alpha, \beta > 0$. Melino and Turnbull [39] also have a model with the asset price being a CEV process but their volatility is a geometric-Ornstein-Uhlenbeck process.

For arbitrary a and b, Hobson [25] reduces the problem to one of optimisation over a set of measures. This optimisation can be performed subject to the solution of a partial differential equation. In the case of Heston [23] he obtains an explicit solution.

We note the work of Jonsson and Sircar [31], where one accumulates wealth as in (8.1), but is concerned with minimising a second moment of the shortfall against hedging a claim based on the same underlying

$$V(x,T) = \inf_{\phi} \mathbb{E}\left[\frac{1}{2}\left(\left(c(P_T) - X_T^{\phi}\right)^+\right)^2\right].$$

That is they have our model of Chapters 2-6 but without the key elements of correlation or a claim revealed through time. In the cases of constant, time-dependent and then *stochastic* volatility they obtain the HJB partial differential equation.

As mentioned above, we obtain an explicit solution to the utility maximisation problem in the case of the second model of Scott [48]. We follow the approach of Hobson [25] and are able to conjecture the form of the required solution to the PDE thereby reducing it to a set of three linked ordinary differential equations. We find explicit solutions to these equations.

8.2 The Dual Problem

We wish to obtain

$$u(x,y,0) = \sup_{\phi} \mathbb{E}\left[U\left(X_T^{\phi}\right) \middle| X_t = x,\, Y_t = y \right]. \tag{8.5}$$

The Lagrangian for the problem at time t is

$$L(X_T^{\phi}, \xi) = \mathbb{E}_t\left[U(X_T^{\phi})\right] - \xi\left(\mathbb{E}_t\left[X_T^{\phi}\frac{d\mathbb{Q}}{d\mathbb{P}}\right] - x\right),$$

where $\mathbb{Q}$ is an equivalent measure under which X is a martingale. Now we have

$$\mathbb{E}_t[U(X_T^{\phi})] = \mathbb{E}_t\left[U(X_T^{\phi}) - \xi\left(X_T^{\phi}\frac{d\mathbb{Q}}{d\mathbb{P}} - x\right)\right] \leq \mathbb{E}_t\left[\tilde{U}\left(\xi\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] + \xi x,$$

where $\tilde{U}$ is the Legendre transform. From (8.4) we have that

$$I(y) = \left(U'\right)^{-1}(y) = y^{-\frac{1}{R}}.$$

In the case $R = 1$, the Legendre transform is given by

$$\tilde{U}(y) = U(I(y)) - yI(y) = -1 - \log y,$$

and so the dual problem has

$$\begin{aligned}
u^D(x,y,0) &= \inf_{\xi}\inf_{\mathbb{Q}}\left\{\xi x - 1 - \log\xi - \mathbb{E}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right]\right\} \\
&= \log x - \sup_{\mathbb{Q}}\mathbb{E}\left[\log\frac{d\mathbb{Q}}{d\mathbb{P}}\right].
\end{aligned}$$

In the case $R \neq 1$, we have

$$\tilde{U}(y) = \frac{R}{1-R}y^{\frac{R-1}{R}},$$

and so

$$\begin{aligned}
u^D(x,y,0) &= \inf_{\xi}\inf_{\mathbb{Q}}\left\{\xi x - \frac{R}{1-R}\xi^{\frac{R-1}{R}}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{\frac{R-1}{R}}\right]\right\} \\
&= \inf_{\xi}\left\{\xi x + \xi^{\frac{R-1}{R}}\inf_{\mathbb{Q}}\left\{-\frac{R}{1-R}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{\frac{R-1}{R}}\right]\right\}\right\}.
\end{aligned}$$

We see that the optimisation over ξ will be straightforward once we have performed the optimisation over $\mathbb{Q}$.

If we write

$$dW_t^2 = \rho dW_t^1 + \rho^\perp dW_t^\perp,$$

so $\rho^\perp = \sqrt{1-\rho^2}$, then admissible state price densities are of the form

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{\int_0^T \left(-\lambda_s\, dW_s^1 - \frac{1}{2}\lambda_s^2\, ds + \psi_s\, dW_s^\perp - \frac{1}{2}\psi_s^2\, ds\right)\right\},$$

for any (ψ_t) satisfying

$$\mathbb{E}\left[\left(\int_0^T \psi_s\, ds\right)^2\right] < \infty,$$

providing

$$\mathbb{E}\left[\left(\int_0^T \lambda_s\, ds\right)^2\right] < \infty.$$

See, for example, Protter [46] p79.

In the case $R=1$ we now have

$$\begin{aligned} u^D(x,y,0) &= U(x) - \sup_\psi \mathbb{E}\left[\int_0^T \left(-\lambda_s\, dW_s^1 + \frac{1}{2}\lambda_s^2\, ds + \psi_s\, dW_s^\perp - \frac{1}{2}\psi_s^2\, ds\right)\right] \\ &= U(x) - \sup_\psi \mathbb{E}\left[\int_0^T \left(\frac{1}{2}\lambda_s^2 - \frac{1}{2}\psi_s^2\right)\, ds\right], \end{aligned}$$

since W_t^1 and $W_t^\perp$ are martingales under $\mathbb{P}$. So

$$u^D(x,y,0) = U(x) - \frac{1}{2}\mathbb{E}\left[\int_0^T \lambda_s^2\, ds\right],$$

since $\psi_t = 0$ is clearly optimal. We see that the case $R=1$ is straightforward and so we now turn our attention to the case $R \neq 1$.

Lemma 8.1 *Suppose we can find* (η_t), $(\eta_t^\perp)$ *and* k *such that*

$$\frac{1}{2}\int_0^T \lambda_s^2\, ds = \int_0^T \left(\eta_s\, dB_s - \eta_s^\perp\, dB_s^\perp + \frac{1}{2}(\eta_s^\perp)^2\, ds - \eta_s^\perp \psi_s\, ds + \frac{1}{2R}(\eta_s - \lambda_s)^2\, ds\right) + k, \tag{8.6}$$

where

$$dB_t = dW_t^1 + \lambda_t dt, \quad dB_t^\perp = dW_t^\perp - \psi_t dt,$$

give independent Brownian motions under $\mathbb{Q}$, then we have

$$u^D(x,y,0) = U(x)e^{-k}.$$

Proof. We have that the state price density satisfies

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{\int_0^T \left(-\lambda_s\, dB_s + \frac{1}{2}\lambda_s^2\, ds + \psi_s\, dB_s^\perp + \frac{1}{2}\psi_s^2\, ds\right)\right\},$$

Using (8.6) this becomes

$$\begin{aligned}\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\Bigg\{ \int_0^T &\Big((\eta_s - \lambda_s)\, dB_s - \frac{1}{2}(q-1)(\eta_s - \lambda_s)^2\, ds \\ &+ (\psi_s - \eta_s^\perp)\, dB_s^\perp + \frac{1}{2}(\psi_s - \eta_s^\perp)^2\, ds\Big) + k\Bigg\},\end{aligned}$$

where $q = (R-1)/R$.

So

$$\begin{aligned}-\frac{1}{q}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right] &= -\frac{1}{q}\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{q-1}\right] \\ &= -\frac{1}{q}\mathbb{E}^{\mathbb{Q}}\Bigg[e^{(q-1)k}\exp\Bigg\{(q-1)\int_0^T (\eta_s - \lambda_s)\, dB_s \\ &\qquad - \frac{1}{2}(q-1)^2\int_0^T (\eta_s - \lambda_s)^2\, ds\Bigg\} \\ &\qquad \times \exp\left\{-\int_0^T \left(\psi_s - \eta_s^\perp\right) dB_s^\perp - \frac{1}{2}\int_0^T \left(\psi_s - \eta_s^\perp\right)^2 ds\right\}^{-(q-1)}\Bigg],\end{aligned}$$

which gives

$$\begin{aligned}-\frac{1}{q}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right] = -\frac{1}{q}e^{(q-1)k}\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\Bigg\{ -\int_0^T \Big(\left(\psi_s - \eta_s^\perp\right) dB_s^\perp \\ - \frac{1}{2}\left(\psi_s - \eta_s^\perp\right)^2 ds\Big)\Bigg\}^{-(q-1)}\Bigg],\end{aligned}$$

Now using Jensen's inequality, since $x \mapsto (-1/q)x^{-(q-1)}$ is convex for $q < 1$ (i.e. $R > 0$),

$$-\frac{1}{q}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right] \geq -\frac{1}{q}e^{(q-1)k}\left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\int_0^T\left(\left(\psi_s - \eta_s^{\perp}\right)dB_s^{\perp}\right.\right.\right.\right.$$
$$\left.\left.\left.\left.-\frac{1}{2}\left(\psi_s - \eta_s^{\perp}\right)^2 ds\right)\right\}\right]\right)^{-(q-1)}.$$

As the above exponential integral is a supermartingale we have

$$-\frac{1}{q}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^q\right] \geq -\frac{1}{q}e^{(q-1)k},$$

with equality for $\psi_t = \eta_t^{\perp}$.

Now

$$\begin{aligned} u^D(x,y,0) &= \inf_{\xi}\left\{\xi x + \inf_{\mathbb{Q}}\left\{-\frac{R}{1-R}\xi^{\frac{R-1}{R}}\mathbb{E}\left[\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^{\frac{R-1}{R}}\right]\right\}\right\}. \\ &= x^{1-R}\left(1 - \frac{R}{1-R}\right)\left(e^{-\frac{k}{R}}\right)^R \\ &= \frac{x^{1-R}}{1-R}e^{-k}, \end{aligned}$$

as required. □

We will find a solution to (8.6) at the end of this section by taking η_t and $\eta_t^{\perp}$ such that both sides of the equation depends only on the process W_t^2.

Heuristically, λ_t gives the market price of trading risk. Now

$$dY_t = a(Y_t)dB_t^2 + \left(b\left(Y_t\right) - \rho a\left(Y_t\right)\lambda_t + \rho^{\perp}a(Y_t)\psi_t\right)dt,$$

where of course $dB_t^2 = \rho dB_t + \rho^{\perp}dB_t^{\perp}$. So $\rho^{\perp}\psi_t - \rho\lambda_t$ gives the market price of volatility risk.

Before solving (8.6) we consider whether any solution would be unique. We find that we do have uniqueness up to certain mild regularity conditions.

Lemma 8.2 *The equation (8.6) has at most one solution with* (η_t) *and* $\left(\eta_t^{\perp}\right)$ *measurable, adapted and satisfying*

$$\mathbb{E}\left[\left(\int_0^T \eta_s\, ds\right)^2\right] < \infty, \quad \mathbb{E}\left[\left(\int_0^T \eta_s^{\perp}\, ds\right)^2\right] < \infty.$$

Proof. Suppose we had

$$\frac{1}{2}\int_0^T \lambda_s^2\,ds = \int_0^T \left(\eta_1\,dB_s - \eta_1^\perp\,dB_s^\perp + \frac{1}{2}(\eta_1^\perp)^2\,ds - \eta_1^\perp\psi\,ds - \frac{1}{2}(q-1)(\eta_1-\lambda_s)^2\,ds\right) + k_1,$$

and

$$\frac{1}{2}\int_0^T \lambda_s^2\,ds = \int_0^T \left(\eta_2\,dB_s - \eta_2^\perp\,dB_s^\perp + \frac{1}{2}(\eta_2^\perp)^2\,ds - \eta_2^\perp\psi\,ds - \frac{1}{2}(q-1)(\eta_2-\lambda_s)^2\,ds\right) + k_2.$$

Subtracting and taking

$$\begin{aligned} dZ_s &= dW_s^1 + \left(\lambda - \frac{1}{2}(q-1)\left(\eta_1+\eta_2-2\lambda_s\right)\right)ds \\ dZ_s^\perp &= dW_s^\perp - \frac{1}{2}\left(\eta_1^\perp + \eta_2^\perp\right)ds, \end{aligned}$$

gives

$$0 = \int_0^T \left(\left(\eta_1-\eta_2\right)dZ_s + \left(\eta_1^\perp - \eta_2^\perp\right)dZ_s^\perp\right) + \left(k_1-k_2\right).$$

Since

$$\mathbb{E}\left[\int_0^T \left(\lambda - \frac{1}{2}(q-1)\left(\eta_1+\eta_2-2\lambda_s\right)\right)^2 ds\right] < \infty,$$

and

$$\mathbb{E}\left[\int_0^T \left(\eta_1^\perp + \eta_2^\perp\right)^2 ds\right] < \infty,$$

Z and $Z^\perp$ are Brownian motions in an equivalent measure. So taking expectations gives $k_1 = k_2$ and then by the uniqueness of martingale representations $\eta_1 = \eta_2$ and $\eta_1^\perp = \eta_2^\perp$, see for example Øksendal [42] p53. □

Consider a slight generalisation of (8.6)

$$\frac{1}{2}\int_t^T \lambda_s^2\,ds = \int_t^T \left(\eta_s\,dB_s - \eta_s^\perp\,dB_s^\perp + \frac{1}{2}(\eta_s^\perp)^2\,ds - \eta_s^\perp\psi_s\,ds - \frac{1}{2}(q-1)(\eta_s-\lambda_s)^2\,ds\right) + k(Y_t,t).$$

Suppose we look for a solution of the form

$$\eta_t = \theta_t \rho,$$
$$\eta_t^{\perp} = -\theta_t \rho^{\perp},$$

this gives

$$\frac{1}{2}\int_t^T \lambda_s^2\, ds = \int_t^T \theta_s\, dW_s^2 + \int_t^T \left(\rho\theta_s\lambda_s + \frac{1}{2}\left(\rho^{\perp}\right)^2 \theta_s^2 - \frac{1}{2}(q-1)\left(\rho\theta_s - \lambda_s\right)^2\right) ds$$
$$+ k(Y_t, t), \qquad (8.7)$$

recalling that

$$dW_t^2 = \rho\left(dB_t - \lambda_t dt\right) + \rho^{\perp}\left(b dB_t^{\perp} + \psi_t dt\right).$$

We notice that the left hand side depends only on W^2 and now the right hand side depends only on W^2. So we have reduced the dimensionality of the problem. Having obtained one solution we can use Lemma 8.2 to conclude that we have the only solution.

Rearranging (8.7), we have

$$\begin{aligned}\frac{1}{2}q\int_t^T \lambda^2\, ds &= \int_t^T \theta_s\, dW_s^2 + \int_t^T \left(q\rho\theta_s\lambda_s + \frac{1}{2}\left(1 - q\rho^2\right)\theta_s^2\right) ds + k(Y_t, t) \\ &= \int_t^T \theta_s\, d\hat{W}_s + \frac{1}{2}\left(1 - q\rho^2\right)\int_t^T \theta_s^2\, ds + k(Y_t, t),\end{aligned}$$

where $d\hat{W}_t = dW_t^2 + q\rho\lambda_t dt$. So,

$$\exp\left\{-\frac{1}{2}pq\int_t^T \lambda^2\, ds\right\} = \exp\left\{-p\int_t^T \theta_s\, d\hat{W}_s - \frac{1}{2}p^2\int_t^T \theta_s^2\, ds\right\} e^{-pk(Y_t,t)},$$

where $p = 1 - q\rho^2$. Taking expectations at time t gives

$$k(Y_t, t) = -\frac{1}{p}\log \hat{\mathbb{E}}\left[\exp\left\{-\frac{1}{2}pq\int_t^T \lambda_s^2\, ds\right\}\right]. \qquad (8.8)$$

Now this probabilistic representation of k is equivalent to

$$k(y, t) = -\frac{1}{p}\log g(y, t),$$

where g satisfies

$$\frac{1}{2}pqg\lambda^2 + \dot{g} + (b - aq\rho\lambda)g' + \frac{1}{2}a^2 g'' = 0, \qquad (8.9)$$

using the Feynman-Kač formula, see for example Øksendal [42] p135.

8.3 The Explicit Solution

We shall consider the model where

$$a(y) = \alpha y, \quad b(y) = \left(\beta + \frac{1}{2}\alpha^2\right) y - \alpha\gamma y \log y,$$

and

$$\lambda(y) = \lambda_0 + \lambda_1 \log y.$$

This form of λ is likely to yield tractable results given that Y, the process driving the volatility, is geometric-Ornstein-Uhlenbeck.

Now (8.9) becomes

$$\begin{aligned}\frac{1}{2}pqg\left(\lambda_0 + \lambda_1 \log y\right)^2 + \dot{g} \\ + \left(\left(\beta + \frac{1}{2}\alpha^2 - \alpha q \rho \lambda_0\right) y - \alpha(\gamma + q\rho\lambda_1) y \log y\right) g' + \frac{1}{2}\alpha^2 g'' = 0.\end{aligned}$$

We simplify this by making the transformation $h(z,t) = g(e^z, t)$ and obtain

$$\frac{1}{2}pq\left(\lambda_0 + \lambda_1 z\right)^2 h + \dot{h} + (\beta - \lambda_0 \alpha q \rho - \alpha(\gamma + q\rho\lambda_1) z) h' + \frac{1}{2}\alpha^2 h'' = 0.$$

Suppose we look for a solution of the form

$$h(z,t) = \exp\left\{A(T-t)z^2 + B(T-t)z + C(T-t)\right\},$$

this gives us

$$\begin{aligned}0 =& z^2\left(-\dot{A} + 2\alpha^2 A^2 - 2A\alpha(\gamma + q\rho\lambda_1) + \frac{1}{2}pq\lambda_1^2\right) \\ &+ z\left(-\dot{B} + 2A(\beta - \alpha q \rho \lambda_0) - \alpha(\gamma + q\rho\lambda_1)B + 2\alpha^2 AB + pq\lambda_0\lambda_1\right) \\ &- \dot{C} + \left(\beta - \alpha q \rho \lambda_0\right) B + \alpha^2 A + \frac{1}{2}\alpha^2 B^2 + \frac{1}{2}pq\lambda_0^2.\end{aligned}$$

Now comparing coefficients gives us a system of three linked ordinary differential equations:

$$\dot{A} = 2\alpha^2 A^2 - 2\alpha\Gamma A + \frac{1}{2}pq\lambda_1^2 \tag{8.10}$$

$$\dot{B} = 2\alpha^2 AB - \alpha\Gamma B + 2\Lambda A + pq\lambda_0\lambda_1 \tag{8.11}$$

$$\dot{C} = \frac{1}{2}\alpha^2 B^2 + \Lambda B + \alpha^2 A + \frac{1}{2}pq\lambda_0^2, \tag{8.12}$$

where $\Gamma = \gamma + q\rho\lambda_1$ and $\Lambda = \beta - \alpha q\rho\lambda_0$.

Proposition 8.1 *We have*

$$A(\tau) = \begin{cases} \frac{1}{2\alpha}\left(\Gamma + \Pi \tan l(\tau)\right) & \text{if } pq\lambda_1^2 > \Gamma^2 \\ \frac{\Gamma^2\tau}{2(\alpha\Gamma\tau+1)} & \text{if } pq\lambda_1^2 = \Gamma^2 \\ \frac{1}{2\alpha}\left(\Gamma + \Pi \tanh l(\tau)\right) & \text{if } pq\lambda_1^2 < \Gamma^2 \end{cases}, \tag{8.13}$$

where $\Pi^2 = \left|\Gamma^2 - pq\lambda_1^2\right|$ *and*

$$l(\tau) = \begin{cases} \alpha\Pi\tau - \tan^{-1}\frac{\Gamma}{\Pi} & \text{if } pq\lambda_1^2 > \Gamma^2 \\ \alpha\Pi\tau - \tanh^{-1}\frac{\Gamma}{\Pi} & \text{if } pq\lambda_1^2 < \Gamma^2 \end{cases}. \tag{8.14}$$

Further,

$$B(\tau) = \begin{cases} \left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)\tan l(\tau) + c_1 \sec l(\tau) - \frac{\Lambda}{\alpha^2} & \text{if } pq\lambda_1^2 > \Gamma^2 \\ \frac{\Lambda\Gamma^2\tau^2}{2(\alpha\Gamma\tau+1)} + pq\lambda_0\lambda_1\frac{\Gamma\alpha\tau^2}{2(\Gamma\alpha\tau+1)} & \text{if } pq\lambda_1^2 = \Gamma^2 \\ \left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)\tanh l(\tau) + c_1 \operatorname{sech} l(\tau) - \frac{\Lambda}{\alpha^2} & \text{if } pq\lambda_1^2 < \Gamma^2 \end{cases}, \tag{8.15}$$

where

$$c_1 = \frac{\Lambda\lambda_1\sqrt{pq}}{\alpha^2\Pi} + \frac{\Gamma\lambda_0}{\sqrt{pq}}.$$

Finally, we have the following expressions for C.

$pq\lambda_1^2 > \Gamma^2$:

$$C(\tau) = \frac{\alpha}{\Pi}\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)c_1\sec l(\tau) + \frac{\alpha}{2\Pi}\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right) + c_1^2\right)\tan l(\tau) \tag{8.16}$$

$$+ \frac{1}{2}\log\sec l(\tau) + \frac{1}{2}\left(\alpha\Gamma + pq\lambda_0^2\right)\tau - c_2,$$

where

$$c_2 = \frac{\alpha}{\Pi}\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)c_1\frac{\lambda_1\sqrt{pq}}{\Pi}$$
$$- \frac{\alpha\Gamma}{2\Pi^2}\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)^2 + c_1^2\right) + \frac{1}{4}\log\frac{pq\lambda_1^2}{\Pi^2}.$$

$pq\lambda_1^2 = \Gamma^2$:

$$C(\tau) = \left(\frac{\tau^3}{12} - \frac{\tau^2}{4\alpha\Gamma} + \frac{3\tau}{4\alpha^2\Gamma^2} - \frac{1}{4\alpha^3\Gamma^3\left(\alpha\Gamma\tau+1\right)} - \frac{\log\left(\alpha\Gamma\tau+1\right)}{\alpha^3\Gamma^3}\right) \tag{8.17}$$
$$\times\left(\Lambda^2\Gamma^2 + 2\Lambda\Gamma\alpha pq\lambda_0\lambda_1 + p^2q^2\lambda_0^2\lambda_1^2\right)$$
$$+\left(\frac{\tau}{4\alpha\Gamma} - \frac{\tau}{2\alpha^2\Gamma^2} + \frac{\log\left(\alpha\Gamma\tau+1\right)}{\alpha^3\Gamma^3}\right)\left(\Lambda^2\Gamma^2 + \Lambda pq\lambda_0\lambda_1\Gamma\alpha\right)$$
$$+\frac{\alpha\Gamma\tau^2}{4} - \frac{1}{2}\tau + \frac{\log(\alpha\Gamma s+1)}{\alpha\Gamma} + \frac{1}{2}pq\lambda_0^2\tau$$
$$+\frac{1}{4\alpha^3\Gamma^3}\left(\Lambda^2\Gamma^2 + 2\Lambda\Gamma\alpha pq\lambda_0\lambda_1 + p^2q^2\lambda_0^2\lambda_1^2\right).$$

$pq\lambda_1^2 < \Gamma^2$:

$$C(\tau) = \frac{\alpha}{\Pi}\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right)c_1\operatorname{sech} l(\tau) + \frac{\alpha}{2\Pi}\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi} + pq\lambda_0\lambda_1\right) + c_1^2\right)\tanh l(\tau) \tag{8.18}$$
$$+\frac{1}{2}\log\operatorname{sech} l(\tau) + \frac{1}{2}\left(\alpha\Gamma + pq\lambda_0^2\right)\tau - c_2.$$

Proof. We see that (8.10) is a separable ODE which can be written as

$$\frac{\dot{A}}{\left(2\alpha A - \Gamma\right)^2 + pq\lambda_1^2 - \Gamma^2} = \frac{1}{2}.$$

First suppose that $pq\lambda_1^2 > \Gamma^2$. Making the transformation

$$2\alpha A - \Gamma = \Pi\tan s$$

gives

$$\int_{-\tan^{-1}\left(\frac{\Gamma}{\Pi}\right)}^{\tan^{-1}\left(\frac{2\alpha A-\Gamma}{\Pi}\right)} \frac{1}{2\alpha\Pi}\,ds = \frac{1}{2}\tau,$$

so

$$\alpha\Pi\tau = \tan^{-1}\left(\frac{2\alpha A-\Gamma}{\Pi}\right) + \tan^{-1}\left(\frac{\Gamma}{\Pi}\right)$$

Rearranging gives the first part of (8.13).

Given A, (8.11) becomes

$$\dot{B} - B\alpha\Pi\tan l(\tau) = \frac{\Lambda}{\alpha}\left(\Gamma + \Pi\tan l(\tau)\right) + pq\lambda_0\lambda_1$$

which is linear with integrating factor

$$\exp\left\{-\int_0^\tau \alpha\Pi\tan\left(\alpha\Pi\, s-\tan^{-1}\left(\frac{\Gamma}{\Pi}\right)\right)\,ds\right\}=\cos l(\tau)+\text{constant}$$

so

$$\begin{aligned}B(\tau)&=\sec l(\tau)\int_0^\tau \cos l(s)\left(\frac{\Lambda}{\alpha}\left(\Gamma+\Pi\tan l(s)\right)+pq\lambda_0\lambda_1\right)\,ds\\&=\sec l(\tau)\left[\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)\sin l(\tau)-\frac{\Lambda}{\alpha^2}\cos l(\tau)\right.\\&\quad\left.+\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)\sin\tan^{-1}\left(\frac{\Gamma}{\Pi}\right)+\frac{\Lambda}{\alpha^2}\cos\tan^{-1}\left(\frac{\Gamma}{\Pi}\right)\right]\end{aligned}$$

Simplifying gives the first part of (8.15).

We can now obtain C from (8.12):

$$\begin{aligned}C(\tau)=\int_0^\tau&\left[\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)\tan l(s)+c_1\sec l(s)-\frac{\Lambda}{\alpha^2}\right)^2\right.\\&+\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)\tan l(s)+c_1\sec l(s)-\frac{\Lambda}{\alpha^2}\right)\\&\left.+\frac{1}{2}\alpha\left(\Gamma+\Pi\tan l(s)\right)+\frac{1}{2}pq\lambda_0^2\right]\,ds.\end{aligned}$$

Gathering together like terms we have

$$\begin{aligned}C(\tau)=\int_0^\tau&\left[\alpha^2\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)c_1\sin l(s)\sec^2 l(s)\right.\\&+\frac{1}{2}\alpha^2\left(\left(\frac{\Gamma\Lambda}{\alpha^2\Pi}+pq\lambda_0\lambda_1\right)^2+c_1^2\right)\sec^2 l(s)\\&\left.+\frac{1}{2}\alpha\Pi\tan l(s)+\frac{1}{2}\left(\alpha\Gamma+pq\lambda_0^2\right)\right]\,ds,\end{aligned}$$

and performing the integration gives (8.16).

The case with $pq\lambda_1^2<\Gamma^2$ is identical but with hyperbolic trigonometric functions replacing circular ones.

Finally suppose that $pq\lambda_1^2=\Gamma^2$. Integrating (8.10) we obtain

$$\frac{1}{2\alpha A-\Gamma}+\frac{1}{\Gamma}=-\alpha\tau,$$

so

$$A(\tau) = \frac{\Gamma\tau}{2\left(\alpha\tau + \frac{1}{\Gamma}\right)},$$

giving the second part of (8.13).

Given A, (8.11) becomes

$$\dot{B} + B\left(\frac{\alpha\Gamma}{\Gamma\alpha\tau + 1}\right) = \frac{\Lambda\Gamma^2\tau}{\Gamma\alpha\tau + 1} + pq\lambda_0\lambda_1,$$

which is linear with integrating factor $\Gamma\alpha\tau + 1$, and so

$$B(\tau) = \frac{\Lambda\Gamma^2}{\Gamma\alpha\tau + 1}\int_0^\tau s\,ds + \frac{pq\lambda_0\lambda_1}{\Gamma\alpha\tau + 1}\int_0^\tau (\Gamma\alpha s + 1)\,ds.$$

which gives the second part of (8.15).

We can now obtain C:

$$\begin{aligned} C(\tau) = \int_0^\tau \Bigg(& \frac{\Gamma^2\alpha^2 s^4}{4(\alpha\Gamma s + 1)^2}\left(\Lambda^2\Gamma^2 + 2\Lambda\Gamma\alpha pq\lambda_0\lambda_1 + p^2q^2\lambda_0^2\lambda_1^2\right) \\ & + \frac{s^2}{2(\alpha\Gamma s + 1)}\left(\Lambda^2\Gamma^2 + \Lambda\Gamma\alpha pq\lambda_0\lambda_1\right) + \frac{\alpha^2\Gamma^2 s}{2(\alpha\Gamma s + 1)} + \frac{1}{2}pq\lambda_0^2\Bigg)\,ds, \end{aligned}$$

which integrates to the required result. □

If we take $\alpha = 0$ then the volatility in our model is actually deterministic. In this case we have, from (8.10)-(8.12),

$$\begin{aligned} u^D(x,y,t) = U(x)\exp\Bigg\{ & \frac{1}{2}q\lambda_1^2(T-t)(\log y)^2 \\ & + \left(\frac{1}{2}\Lambda q\lambda_1^2(T-t)^2 + q\lambda_0\lambda_1(T-t)\right)\log y \\ & + \frac{1}{6}\Lambda^2 q\lambda_1^2(T-t)^3 + \frac{1}{2}\Lambda q\lambda_0\lambda_1(T-t)^2 + \frac{1}{2}q\lambda_0^2(T-t)\Bigg\}, \end{aligned}$$

which rearranges to give

$$u^D(x,y,t) = U(x)\exp\left\{\frac{1}{2}q(T-t)\lambda(y)^2 + \frac{1}{2}q\Lambda\lambda_1(T-t)^2\lambda(y) + \frac{1}{6}q\Lambda^2\lambda_1^2(T-t)^3\right\}.$$

We notice that this is jointly homogeneous in $(T-t)$ and $\lambda(y)$.

8.4 The Primal Problem

Consider again the original primal problem (8.5). We can write this as

$$u(X_t, Y_t, t) = \sup_{\phi} \mathbb{E}_t \left[u\left(X_t^{\phi} + dX_t^{\phi},\, Y_t + dY_t,\, t + dt \right) \right].$$

We use the approach we used in Chapter 2, in obtaining the HJB equation for our perfect hedging problem, which is essentially the approach used by Zariphopoulou [53]. The dynamic programming equation is

$$0 = \sup_{\phi} \left\{ \frac{1}{2} u_{xx} \phi^2 x^2 y^2 + \phi x y \left(u_x \lambda + u_{xy} a \rho \right) \right\} + u_y b + \frac{1}{2} u_{yy} a^2 + \dot{u}. \tag{8.19}$$

We find that the optimal policy is

$$\phi^* = -\frac{u_x \lambda + u_{xy} a \rho}{x y u_{xx}},$$

and that the value function satisfies

$$0 = u_y b + \frac{1}{2} u_{yy} a^2 + \dot{u} - \frac{\left(u_x \lambda + u_{xy} a \rho \right)^2}{2 u_{xx}}. \tag{8.20}$$

Suppose we conjecture that there is a solution to (8.20) of the form

$$u(x, e^z, t) = U(x) \left(v(z, t) \right)^{\delta},$$

then we obtain

$$0 = b v' v^{\delta-1} \delta + \frac{1}{2} a^2 \delta \left(v'' v^{\delta-1} + (\delta - 1) \left(v' \right)^2 v^{\delta-2} \right) + \dot{v} v^{\delta-1} \delta - q \frac{\left(v^{\delta} \lambda + \delta v' v^{\delta-1} a \rho \right)^2}{2 v^{\delta}},$$

which becomes

$$0 = v' b + \frac{1}{2} a^2 v'' + \dot{v} + \frac{1}{2} (\delta - 1) a^2 \frac{\left(v' \right)^2}{v} - \frac{q}{2\delta} v \lambda^2 - q a \rho \lambda v' - \frac{1}{2} \delta q \rho \frac{\left(v' \right)^2}{v} a^2,$$

Taking $\delta = 1/p$ we recover the partial differential equation (8.9) we had for the dual problem.

In the geometric-Ornstein-Uhlenbeck case that we have been considering we find

$$\begin{aligned}\phi^* =& \frac{-R}{-x^{-R-1}}\left(\log y x^{-R} + \rho\alpha y x^{-R}\frac{1}{p}\left(A(T-t)2\frac{\log y}{y} + B(T-t)\frac{1}{y}\right)\right)\\ =& R\left(\log y + \frac{\rho\alpha}{p}\left(2A(T-t)\log y + B(T-t)\right)\right).\end{aligned}$$

We notice that this is independent of the current wealth level x. We would expect this given that our utility (8.4) is separable.

The key advantage that the dual approach offers is that it does not need an assumption that the processes involved are Markovian until after Lemma 8.2. It also gives the pricing measure which one would use for utility-indifference pricing in this model. This would not be straightforward to obtain by the primal approach.

Bibliography

[1] C. Acerbi and D. Tasche. On the coherence of expected shortfall. Technische Universität München Working Paper, 2001.

[2] P. Artzner, F. Delbaen, J-M. Eber, and D. Heath. Coherent measures of risk. *Mathematical Finance*, 3:203–228, 1999.

[3] C. A. Ball and A. Roma. Stochastic volatility option pricing. *Journal of Financial and Quantitative Analysis*, 29:589–607, 1994.

[4] S. Browne. Optimal investment policies for a firm with a random risk process: exponential utility and minimzing the probability of ruin. *Mathematics of Operational Research*, 20:937–958, 1995.

[5] S. Browne. Survival and growth with a liability: optimal portfolio strategies in continuous time. *Mathematics of Operational Research*, 22:468–493, 1997.

[6] S. Browne. Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Advances in Applied Probability*, 31:551–577, 1999.

[7] S. Browne. Risk-constrained dynamic active portfolio management. *Management Science*, 46:1188–1199, 2000.

[8] J. C. Cox, J. E. Ingersoll, and S. A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53:385–407, 1985.

[9] C. Donati-Martin, R. Ghomrasni, and M. Yor. On certain markov processes attached to exponential functionals of brownian motion; application to asian options. *Revista Matematica Iberoamericana*, 17:179–193, 2001.

[10] K. Dowd. Sharpe thinking. *Risk*, 14:S22–S24, 2001.

[11] D. Dufresne. An affine property of the reciprocal Asian option process. *Osaka Journal of Mathematics*, 38:379–381, 2001.

[12] D. Dufresne. The integral of geometric Brownian motion. *Advances in Applied Probability*, 33:223–241, 2001.

[13] L. C. Evans. *Partial Differential Equations.* AMS, 1998.

[14] G. Favero and T. Vargiolu. Robustness of shortfall risk minimising strategies in the binomial model. Università degli Studi di Padova Working Paper, 2002.

[15] T. S. Ferguson. Betting systems which minimize the probability of ruin. *Journal of SIAM*, 13:795–818, 1965.

[16] H. Föllmer and P. Leukert. Quantile hedging. *Finance and Stochastics*, 3:251–273, 1999.

[17] H. Föllmer and P. Leukert. Efficient hedging: Cost versus shortfall risk. *Finance and Stochastics*, 4:117–146, 2000.

[18] R. Frey. Derivative asset analysis in models with level-dependent and stochastic volatility. *CWI Quarterly*, 10:1–34, 1997.

[19] A. Friedman. *Partial Differential Equations of Parabolic Type.* Prentice Hall, 1964.

[20] D. Heath. A continuous time version of Kulldorff's result. Unpublished Manuscript, 1993.

[21] D. Heath, S. Orey, V. Pestien, and W. Sudderth. Minimizing or maximizing the expected time to reach zero. *SIAM Journal of Control and Optimization*, 25:195–205, 1987.

[22] V. Henderson and R. Wojakowski. On the equivalence of floating- and fixed-strike Asian options. *Journal of Applied Probability*, 39:391–394, 2002.

[23] S. L. Heston. A closed form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[24] D. G. Hobson. Stochastic volatility. In D. J. Hand and S. D. Jacka, editors, *Statistics in Finance*, pages 284–306. Arnold, 1998.

[25] D. G. Hobson. Stochastic volatility models, correlation and the q-optimal measure. University of Bath Preprint, 2002.

[26] S. Hodges. A generealization of the Sharpe ratio and its applications to valuation bounds and risk measures. FORC Preprint 88, 1998.

[27] J. C. Hull. *Options, Futures and Other Derivatives.* Prentice-Hall, 1997.

[28] J. C. Hull and A. White. The pricing of options on assets with stochastic volatilities. *The Journal of Finance*, 42:281–300, 1987.

[29] J. C. Hull and A. White. An analysis of the bias in option pricing caused by a stochastic volatility. *Advances in Futures and Options Research*, 3:29–61, 1988.

[30] H. Johnson and D. Shanno. Option pricing when the variance is changing. *Journal of Financial and Quantitative Analysis*, 22:143–151, 1987.

[31] M. Jonsson and K. R. Sircar. Partial hedging in a stochastic volatility environment. *Mathematical Finance*, 12:375–409, 2002.

[32] I. Karatzas. Adaptive control of a diffusion to a goal and a parabolic Monge-Ampère-type equation. *Asian Journal of Mathematics*, 1:295–313, 1997.

[33] I. Karatzas. Probabilistic aspects of portfolio analysis and optimization. Second World Congress of Bachelier Finance Society, 2002.

[34] N. El Karoui and M-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal of Control and Optimization*, 33:29–66, 1995.

[35] J. L. Kelly. A new interpretation of information rate. *Bell Systems Technical Journal*, 35:917–926, 1956.

[36] M. Kulldorff. Optimal control of favourable games with a time limit. *SIAM Journal of Control and Optimization*, 31:52–69, 1993.

[37] M. Majumder and R. Radner. Linear models of economic survival under production uncertainty. *Economic Theory*, 1:13–30, 1991.

[38] B. Mandelbrot. The variation of certain speculative prices. *Journal of Business*, 36:394–419, 1963.

[39] A. Melino and S. M. Turnbull. Pricing foreign currency options with stochastic volatility. *Journal of Econometrics*, 45:239–265, 1990.

[40] R. Merton. An intertemporal capital asset pricing model. *Econometrica*, 41:867–887, 1973.

[41] M. A. Milevsky and S. E. Posner. Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of Financial and Quantitative Analysis*, 33:409–422, 1998.